CLASSICAL GROUPS

By HERMANN WEYL

THE CLASSICAL GROUPS

THEIR INVARIANTS AND REPRESENTATIONS

BY

HERMANN WEYL

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PREFACE TO THE FIRST EDITION

Ever since the year 1925, when I succeeded in determining the characters of the semi-simple continuous groups by a combination of E. Cartan's infinitesimal methods and I. Schur's integral procedure, I have looked toward the goal of deriving the decisive results for the most important of these groups by direct algebraic construction, in particular for the full group of all non-singular linear transformations and for the orthogonal group. Owing mainly to R. Brauer's intervention and collaboration during the last few years, it now appears that I have in my hands all the tools necessary for this purpose. The task may be characterized precisely as follows: with respect to the assigned group of linear transformations in the underlying vector space, to decompose the space of tensors of given rank into its irreducible invariant subspaces. In other words, our concern is with the various kinds of "quantities" obeying a linear transformation law, which may be prepared under the reign of each group from the material of tensors. Such is the problem which forms one of the mainstays of this book, and in accordance with the algebraic approach its solution is sought for not only in the field of real numbers on which analysis and physics fight their battles, but in an arbitrary field of characteristic zero. However, I have made no attempt to include fields of prime characteristic.

The notion of an algebraic invariant of an abstract group γ cannot be formulated until we have before us the concept of a representation of γ by linear transformations, or the equivalent concept of a "quantity of type $\mathfrak{A}$" The problem of finding all representations or quantities of γ must therefore logically precede that of finding all algebraic invariants of γ. (For the notion of quantities and invariants of a more general character, and their close interdependence, the reader is referred to the restatement in Chapter I of Klein's Erlanger program in slightly more abstract terms.) My second aim, then, is to give a modern introduction to the theory of invariants. It is high time for a rejuvenation of the classic invariant theory, which has fallen into an almost petrified state. My vindication for having proceeded in a much more conservative manner than our young generation of algebraists would probably deem desirable, is the wish not to sacrifice the past; even so, I hope to have broken through to the modern concepts resolutely enough. I do not pretend to have written *the* book on modern invariant theory: A systematic handbook would have to include many things passed over in silence here.

As one sees from the above description, the subject of this book is rather special. Important though the general concepts and propositions may be with which the modern industrious passion for axiomatizing and generalizing has presented us, in algebra perhaps more than anywhere else, nevertheless I am convinced that the special problems in all their complexity constitute the stock and core of mathematics; and to master their difficulties requires on the whole the harder labor. The border line is of course vague and fluctuating. But quite intentionally scarcely more than two pages are devoted to the general theory of group representations, while the application of this theory to the particular groups that come under consideration occupies at least fifty times as much space. The general theories are shown here as springing forth from special problems whose analysis leads to them with almost inevitable necessity as the fitting tools for their solution; once developed, these theories spread their light over a wide region beyond their limited origin. In this spirit we shall treat among others the doctrine of associative algebras, which in the last decade has risen to a ruling position in mathematics.

The relations to other parts of mathematics are emphasized where occasion arises, and despite the fundamentally algebraic character of the book, neither the infinitesimal nor the topological methods have been omitted. My experience has seemed to indicate that to meet the danger of a too thorough specialization and technicalization of mathematical research is of particular importance in America. The stringent precision attainable for mathematical thought has led many authors to a mode of writing which must give the reader an impression of being shut up in a brightly illuminated cell where every detail sticks out with the same dazzling clarity, but without relief. I prefer the open landscape under a clear sky with its depth of perspective, where the wealth of sharply defined nearby details gradually fades away towards the horizon. In particular, the massif of topology lies for this book and its readers at the horizon, and hence what parts of it had to be taken into the picture are given in broad outline only. An adaptation of sight different from that required in the algebraic parts, and a sympathetic willingness to cooperate, are here expected from the reader.

The book is primarily meant for the humble who want to learn as new the things set forth therein, rather than for the proud and learned who are already familiar with the subject and merely look for quick and exact information about this or that detail. It is neither a monograph nor an elementary textbook. The references to the literature are handled accordingly.

The gods have imposed upon my writing the yoke of a foreign tongue that was not sung at my cradle.

> "Was dies heissen will, weiss jeder,
> Der im Traum pferdlos geritten,"

I am tempted to say with Gottfried Keller. Nobody is more aware than myself of the attendant loss in vigor, ease and lucidity of expression. If at least the worst blunders have been avoided, this relative accomplishment is to be ascribed solely to the devoted collaboration of my assistant, Dr. Alfred H. Clifford; and even more valuable for me than the linguistic, were his mathematical criticisms.

HERMANN WEYL

PRINCETON, N. J.,
September, 1938.

NOTE. A reference to formula (7.6) [or to (3.7.6)] indicates the formula 6 in section 7 labeled as (7.6) in the *same* chapter [or in Chapter III respectively].

PREFACE TO THE SECOND EDITION

The photostatic process employed for the reprinting ruled out any appreciable changes which otherwise might have been desirable. But a new chapter containing Supplements, a list of Errata and Addenda, and a short Bibliography for the years 1940–1945 have been added. Two of the supplements develop an alternate and more direct approach to some of the problems in the theory of the orthogonal and symplectic groups dealt with in Chapters II, V and VI. Supplement C describes a particularly straightforward and powerful process for the generation of invariants discovered by M. Schiffer, whereas supplement D applies the "matrix method" of Chapters III and IX to the splitting of a division algebra by extension of the ground field, without the limitation to normal algebras and finite extensions.

HERMANN WEYL

PRINCETON, N. J.,
March, 1946.

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CHAPTER I

INTRODUCTION

1. Fields, rings, ideals, polynomials

Before we can start talking algebra we must fix the *field* k of numbers wherein we operate. k is the closed universe in which all our actions take place. I should advise the reader at first to think of k as the continuum of the ordinary real or complex numbers. But generally speaking, k is any set of elements α, called *numbers*, closed with respect to the two binary operations: *addition* and *multiplication*. Addition and multiplication are supposed to be *commutative* and *associative*. Moreover, addition shall allow of a unique inversion (*subtraction*), i.e. there is a number o, called *zero*, such that

$$\alpha + o = \alpha$$

for every α, and each α has a *negative* $-\alpha$ satisfying $\alpha + (-\alpha) = o$. Multiplication shall fulfill the *distributive law* with respect to addition:

$$\alpha(\beta + \gamma) = (\alpha\beta) + (\alpha\gamma),$$

from which one readily deduces the universal equation

$$\alpha \cdot o = o. \tag{1.1}$$

Multiplication also is required to be invertible (*division*) with the one exception necessarily imposed by (1.1): there shall exist a *unit* ϵ or 1 satisfying

$$\alpha \cdot \epsilon = \alpha \tag{1.2}$$

for all α, and every α *except* o shall have an *inverse* α^{-1} or $1/\alpha$ such that $\alpha \cdot \alpha^{-1} = \epsilon$. Were $\epsilon = o$, all numbers α would be $= o$ according to (1.1) and (1.2); this degenerate case we once for all exclude by the axiom $\epsilon \neq o$.

Any number α gives rise to its *multiples*

$$\alpha = 1\alpha, \qquad \alpha + \alpha = 2\alpha, \qquad 2\alpha + \alpha = 3\alpha, \cdots;$$

here the integers 1, 2, 3, $\cdots$ are symbols of "multipliers" rather than numbers in the reference field k. Two cases are possible: either all the multiples

$$n\epsilon \qquad (n = 1, 2, 3, \cdots)$$

of the unit ϵ are $\neq o$, or there is a least n for which $n\epsilon = o$. In the latter case the integer n must be a prime number p. Indeed for a composite number $n = n_1n_2$ (neither n_1 nor $n_2 = 1$) we should have

$$n\epsilon = n_1\epsilon \cdot n_2\epsilon = o,$$

and hence $n_1\epsilon$ or $n_2\epsilon$ would equal o in contradiction to n being the *least* vanishing multiple of ϵ. One distinguishes these two cases by ascribing the *characteristic* 0 *or* p to the field k. In a field of prime characteristic p the p-fold of any number α vanishes:

$$p\alpha = p(\epsilon\alpha) = (p\epsilon)\alpha = o.$$

In a field of characteristic zero we can form the aliquot part $\beta = \alpha/n$ of α with any integer n, i.e. a number β satisfying the equation $n\beta = \alpha$. Indeed this equation amounts to

$$n\epsilon\cdot\beta = \alpha,$$

and as the first factor $n\epsilon$ is $\neq o$ the equation is solvable according to the axiom of divisibility. Hence our field k contains the subfield of the rational multiples of ϵ:

$$m\epsilon/n \quad (n \text{ a positive integer } 1, 2, 3, \cdots,$$

$$m \text{ any integer } 0, \pm 1, \pm 2, \cdots),$$

which is isomorphic to the field of ordinary rational numbers m/n and may be identified with it. To this most primitive field of characteristic 0 we shall always refer as the *ground field* κ, and our remark thus asserts the fact that *any field k of characteristic* 0 *contains the ground field κ. From now on we shall assume the reference field k to be of characteristic* 0 without mentioning this restriction again and again; we shall not try to discuss any of our problems in a field of prime characteristic. So even when we use the phrase "in an arbitrary field" or something similar we mean "in an arbitrary field of characteristic 0".

If one omits the axiom requiring the existence of an inverse α^{-1} one obtains the general notion of a *ring* rather than a field; only addition, subtraction and multiplication are possible in a ring. The classical example is the set of all integers. If a product $\alpha\beta$ of two elements of the ring never vanishes unless at least one of the factors vanishes, the ring is *without null divisors*. Starting with a given ring R without null divisors, we may formally introduce fractions α/β as pairs of elements α, β in R of which the second term β is $\neq 0$, and then define equality, addition, and multiplication in accordance with the rules which we all learned in school. The fractions form a field, the *quotient field of R*; it contains R if we identify the fraction $\alpha/1$ with α.

With respect to a given ring R a set $\mathfrak{a}$ of its elements is called an *ideal* if

$$\alpha \pm \beta, \qquad \lambda\alpha$$

lie in $\mathfrak{a}$ for any α, β in $\mathfrak{a}$ and any number λ in R. The case where $\mathfrak{a}$ consists of the one element 0 only is expressly excluded. The classical example is provided by the integral multiples of a given integer. The ideals serve as modules for *congruences*:

$$\lambda \equiv \mu \pmod{\mathfrak{a}}$$

means that the difference $\lambda - \mu$ of the two numbers λ, μ of R lies in $\mathfrak{a}$. A finite number of elements $\alpha_1, \cdots, \alpha_r$ in $\mathfrak{a}$ constitute an (ideal) basis of $\mathfrak{a}$ if every element α in $\mathfrak{a}$ is of the form

$$\lambda_1\alpha_1 + \cdots + \lambda_r\alpha_r \qquad (\lambda_i \text{ in } R).$$

$\mathfrak{a}$ is then the ideal $(\alpha_1, \cdots, \alpha_r)$ with the basis $\alpha_1, \cdots, \alpha_r$. In a *field* k there is only one ideal, the field itself. For if α is a number $\neq 0$ in the given ideal $\mathfrak{a}$, the latter will contain all numbers of the form $\lambda\alpha$ and hence every number β whatsoever: $\lambda = \beta\alpha^{-1}$. In the ring of ordinary integers every ideal is a principal ideal (α). $\mathfrak{a}$ is a *prime ideal* if the congruence

$$\lambda\mu \equiv 0 \pmod{\mathfrak{a}}$$

never holds unless one of the factors λ, μ is $\equiv 0 \pmod{\mathfrak{a}}$.

A formal expression

$$f(x) = \sum_{i=0}^{n} \alpha_i x^i$$

involving the "indeterminate" (or variable) x, whose coefficients α_i are numbers in a field k, is called a (k-)polynomial of x of formal degree n. If $\alpha_n \neq 0$, n is its actual degree; 0 is the only polynomial not possessing an actual degree. Everybody knows how to add and multiply polynomials; they form a ring $k[x]$ without null-divisors. Indeed if a is of actual degree m, b of degree n:

$$a = \alpha_m x^m + \cdots, \quad b = \beta_n x^n + \cdots \qquad (\alpha_m \neq 0, \beta_n \neq 0),$$

then

$$ab = \alpha_m\beta_n x^{m+n} + \cdots$$

is of degree $m + n$ since $\alpha_m\beta_n \neq 0$. One sees that this proposition will still hold when the coefficients are taken from a ring without null-divisors rather than from a field k. This allows us to pass to polynomials of a new indeterminate y with coefficients taken from $k[x]$ or, what is the same, to k-polynomials of *two* indeterminates x, y, and so on. *The k-polynomials of several indeterminates $x, y, \cdots$ form a ring $k[x, y, \cdots]$ without null-divisors.*

In a given polynomial $F(u, v, \cdots)$ of certain indeterminates $u, v, \cdots$ one may carry out the *substitution*

$$u = f(x, y, \cdots), \qquad v = g(x, y, \cdot\cdot), \cdots$$

by means of certain polynomials $f, g, \cdots$ of other indeterminates $x, y, \cdots$; the result is a polynomial $\Phi(x, y, \cdots)$ of $x, y, \cdots$:

$$F(f(x, y, \cdots), g(x, y, \cdots), \cdots) = \Phi(x, y, \cdots).$$

In particular one may substitute *numbers* $\alpha, \beta, \cdots$ for the "arguments" $u, v, \cdots$ in F; the resulting number $F(\alpha, \beta, \cdots)$ is called the *value of F for the values $\alpha, \beta, \cdots$ of the arguments $u, v, \cdots$.*

$f(x)$ being a polynomial in x, α is a *zero* or a *root* of f if $f(\alpha) = 0$. A polynomial of degree n has at most n different zeros; this follows in the well-known way by proving that $f(x)$ contains the factors $(x - \alpha_1)(x - \alpha_2) \cdots$ if $\alpha_1, \alpha_2, \cdots$ are distinct zeros. Hence a polynomial $f(x) \neq 0$ does not vanish numerically for every value of x in k, provided the reference field k is of characteristic 0, because such a field contains infinitely many numbers. One can even find a *rational* value of x for which the value of f is $\neq 0$. Induction with respect to the number of indeterminates permits generalization of this proposition to polynomials with any number of arguments. If

$$F(x, y, \cdots); \qquad R_1(x, y, \cdots), \qquad R_2(x, y, \cdots), \cdots$$

are a number of non-vanishing k-polynomials then the product $FR_1R_2 \cdots$ is also $\neq 0$; and hence our statement can be sharpened to the following

LEMMA (1.1.A). (*Principle of the irrelevance of algebraic inequalities.*) *A k-polynomial $F(x, y, \cdots)$ vanishes identically if it vanishes numerically for all sets of rational values $x = \alpha$, $y = \beta$, $\cdots$ subject to a number of algebraic inequalities*

$$R_1(\alpha, \beta, \cdots) \neq 0, \qquad R_2(\alpha, \beta, \cdots) \neq 0, \cdots.$$

From the ring $k[x, y, \cdots]$ of k-polynomials in $x, y, \cdots$ one can pass to the *field* $k(x, y, \cdots)$ of the *rational functions* of $x, y, \cdots$ in k by forming the quotient field of $k[x, y, \cdots]$.

The *derivative* $f'(x)$ of a polynomial $f(x)$ is introduced as the coefficient of t in the expansion of $f(x + t)$ as a polynomial in t:

$$f(x + t) = f(x) + t \cdot f'(x) + \cdots \tag{1.3}$$

The familiar formal properties of derivation are immediate consequences thereof. One might restate the definition (1.3) as follows: there is a polynomial $g(x, y)$ satisfying the identity

$$f(y) - f(x) = (y - x) \cdot g(x, y); \tag{1.4}$$

$f'(x)$ is $= g(x, x)$. While in Calculus the unique determination of $g(x, x)$ is brought about by requiring $g(x, y)$ to be *continuous* even for $y = x$, Algebra attains the same by requiring g to be a *polynomial*. The derivative of

$$f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$$

is

$$f'(x) = \alpha_1 + 2\alpha_2 x + \cdots + n\alpha_n x^{n-1}.$$

Hence *the only polynomial $f(x)$ in a field of characteristic zero whose derivative $f'(x)$ vanishes is the constant: $f(x) = \alpha_0$.*

For a polynomial $f((x)) = f(x_1, \cdots, x_n)$ of n variables x_i one may form similarly to (1.3):

$$f((x + t \cdot y)) = f(x_1 + ty_1, \cdots, x_n + ty_n) = f((x)) + t \cdot f_1((x, y)) + \cdots. \tag{1.5}$$

The coefficient $f_1((x, y))$ of t in this expansion by t is called the *polarized polynomial* $D_{yx}f$ of f; it involves the new variables y_i in a homogeneous linear fashion:

$$D_{yx}f = \frac{\partial f}{\partial x_1} y_1 + \cdots + \frac{\partial f}{\partial x_n} y_n. \tag{1.6}$$

Sometimes the new variables y_i are designated by dx_i and then the polarized form is called the *total differential* df of f. The polar process has the formal properties of differentiation:

$$\begin{aligned} D(f + g) &= Df + Dg, \\ D(\alpha f) &= \alpha \cdot Df \quad (\alpha \text{ a number}), \\ D(f \cdot g) &= Df \cdot g + f \cdot Dg. \end{aligned} \tag{1.7}$$

The degree of a *monomial*

$$x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$$

of our n variables $x_1, x_2, \cdots, x_n$ is the sum

$$r = r_1 + r_2 + \cdots + r_n$$

of the non-negative integral exponents $r_1, \cdots, r_n$. Each polynomial $f((x))$ is a linear combination of monomials; if all these monomials are of the same degree r:

$$f((x)) = \sum \alpha_{r_1 \cdots r_n} x_1^{r_1} \cdots x_n^{r_n}, \qquad (r_1 + \cdots + r_n = r) \tag{1.8}$$

the polynomial is called *homogeneous* or *a form of degree r*. In (1.8) the sum extends over all sets of non-negative integral exponents $r_1, \cdots, r_n$ with the sum r. Multiplication of all variables x_i with a numerical factor λ has the effect of changing

$$f((x)) \quad \text{into} \quad \lambda^r \cdot f((x)). \tag{1.9}$$

Another way of writing such a form is this:

$$f((x)) = \sum_{i=1}^{n} \beta(i_1, \cdots, i_r) x_{i_1} \cdots x_{i_r} \tag{1.10}$$

where each of the r indices i runs independently from 1 to n. In this expression the coefficients β are not uniquely determined; they become so, however, if one imposes the condition of symmetry upon the β:

$$\beta(i_{1'}, \cdots, i_{r'}) = \beta(i_1, \cdots, i_r)$$

provided $1', \cdots, r'$ is any permutation of $1, \cdots, r$. Then the β are obviously linked to the α's by the following relation:

$$\alpha_{r_1 \cdots r_n} = \frac{r!}{r_1! \cdots r_n!} \beta(i_1, \cdots, i_r) \tag{1.11}$$

if r_1 of the r indices i_α are $= 1$, r_2 of them $= 2$, $\cdots$, r_n of them $= n$.

(1.10) suggests introducing the multilinear form

$$f((x, y, \cdots, z)) = \sum \beta(i_1 i_2 \cdots i_r) x_{i_1} y_{i_2} \cdots z_{i_r} \tag{1.12}$$

depending on r sets of n variables:

$$\begin{aligned} x &= (x_1, \cdots, x_n), \\ y &= (y_1, \cdots, y_n), \\ &\cdots\cdots\cdots\cdots \\ z &= (z_1, \cdots, z_n). \end{aligned} \tag{1.13}$$

From it we fall back upon the form (1.10) by identifying

$$x = y = \cdots = z \quad \text{with} \quad x. \tag{1.14}$$

Symmetry of the coefficients β with respect to the indices i_α is equivalent to the symmetry of the multilinear form $f(x, y, \cdots, z)$ with respect to permutation of the r sets $x, y, \cdots, z$. Hence our result may be expressed thus: there exists a uniquely determined symmetric multilinear form $f(x, y, \cdots, z)$ which by the identification (1.14) passes into a given form $f((x))$ of degree r.

On putting $\lambda = 1 + t$ in (1.9) one finds that the polarized form $D_{yx}f$ changes back into $r \cdot f$ if y is replaced by x:

$$\{D_{yx}f(x)\}_{y=x} = r \cdot f(x).$$

The same is clear from (1.10) which, under the assumption of symmetric β's, at once yields

$$D_{yx}f = r \cdot \sum_{i_1, \cdots, i_r} \beta(i_1, i_2, \cdots, i_r) y_{i_1} x_{i_2} \cdots x_{i_r}.$$

Hence the symmetric multilinear form $f(x, y, \cdots, z)$ corresponding to the given form $f(u)$ of degree r arises from $f = f(u)$ by *complete polarization*:

$$D_{xu} D_{yu} \cdots D_{zu} f(u) = r! \sum \beta(i_1, i_2, \cdots, i_r) x_{i_1} y_{i_2} \cdots z_{i_r}.$$

This again shows its uniqueness.

2. Vector space

The next fundamental concept on which we must come to a common understanding right at the beginning is that of *vector space* (*in k*). A vector space P is a *k-linear set* of elements, called vectors; i.e. a domain in which addition of vectors and multiplication of a vector by a number in k are the permissible operations, satisfying the well-known rules of vector geometry.[1] n vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ form a *coordinate system* or a *basis* if they are linearly independent, while enlargement of the sequence by any further vector would destroy this independence. Under these assumptions every vector $\mathfrak{x}$ is uniquely expressible in the form

$$\mathfrak{x} = x_1 \mathfrak{e}_1 + \cdots + x_n \mathfrak{e}_n \tag{2.1}$$