Texts and Monographs in Physics

Norbert Straumann

General Relativity and Relativistic Astrophysics

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With 81 Figures

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Texts and
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To my wife and sons,
Maria, Dominik, Felix, and Tobias

Preface

In 1979 I gave graduate courses at the University of Zurich and lectured in the 'Troisième Cycle de la Suisse Romande' (a consortium of four universities in the french-speaking part of Switzerland), and these lectures were the basis of the 'Springer Lecture Notes in Physics', Volume 150, published in 1981. This text appeared in German, because there have been few modern expositions of the general theory of relativity in the mother tongue of its only begetter. Soon after the book appeared, W. Thirring asked me to prepare an English edition for the 'Texts and Monographs in Physics'. Fortunately E. Borie agreed to translate the original German text into English. An excellent collaboration allowed me to revise and add to the contents of the book. I have updated and improved the original text and have added a number of new sections, mostly on astrophysical topics. In particular, in collaboration with M. Camenzind I have included a chapter on spherical and disk accretion onto compact objects.

This book divides into three parts. Part I develops the mathematical tools used in the general theory of relativity. Since I wanted to keep this part short, but reasonably self-contained, I have adopted the dry style of most modern mathematical texts. Readers who have never before been confronted with differential geometry will find the exposition too abstract and will miss motivations of the basic concepts and constructions. In this case, one of the suggested books in the reference list should help to absorb the material. I have used notations as standard as possible. A collection of important formulae is given at the end of Part I. Many readers should start there and go backwards, if necessary.

In the second part, the general theory of relativity is developed along rather traditional lines. The coordinate-free language is emphasized in order to avoid unnecessary confusions. We make full use of Cartan's calculus of differential forms which is often far superior computationally. The tests of general relativity are discussed in detail and the binary pulsar PSR 1913+16 is fully treated.

The last part of the book treats important aspects of the physics of compact objects. Some topics, for example the cooling of neutron stars, are discussed in great detail in order to illustrate how astrophysical problems require the simultaneous application of several different disciplines.

A text-book on a field as developed and extensive as general relativity and relativistic astrophysics must make painful omissions. Since the emphasis throughout is on direct physical applications of the theory, there is little discussion of more abstract topics such as causal spacetime structure or singularities. Cosmology, which formed part of the original lectures, has been omitted entirely. This field has grown so much in recent years that an entire book should be devoted to it. Furthermore, Weinberg's book still gives an excellent introduction to the more established parts of the subject.

The reference list near the end of the book is confined to works actually cited in the text. It is certainly much too short. In particular, we have not cited the early literature of the founders. This is quoted in the classic article by W. Pauli and in the wonderful recent book of A. Pais 'Subtle is the Lord', which gives also a historical account of Einstein's struggle to general relativity.

The physics of compact objects ist treated more fully in a book by S. L. Shapiro and S. A. Teukolsky, which just appeared when the final pages of the present English edition were typed.

I thank E. Borie for the difficult job of translating the original German text and her fine collaboration. I am particularly grateful to M. Camenzind for much help in writing the chapter on accretion and to J. Ehlers for criticism and suggestions for improvements. I profited from discussions and the writings of many colleagues. Among others, I am indepted to G. Boerner and W. Hillebrandt. R. Durrer, M. Schweizer, A. Wipf and R. Wallner helped me to prepare the final draft. I thank D. Oeschger for her careful typing of the German and English manuscripts.

For assistance in the research that went into this book, I thank the Swiss National Science Foundation for financial support.

Finally I thank my wife Maria for her patience.

Zurich, July 1984 *N. Straumann*

Contents

Part I

Differential Geometry

In this purely mathematical part, we develop the most important concepts and results of differential geometry which are needed for general relativity theory.

The presentation differs little from that in many contemporary mathematical text books (however, some topics, such as fiber bundles, will be omitted). The language of modern differential geometry and the "intrinsic" calculus on manifolds are now frequently used by workers in the field of general relativity and are beginning to appear in textbooks on the subject. This has a number of advantages, such as:

(i) It enables one to read the mathematical literature and make use of the results to attack physical problems.

(ii) The fundamental concepts, such as differentiable manifolds, tensor fields, affine connection, and so on, adopt a clear and intrinsic formulation.

(iii) Physical statements and conceptual problems are not confused by the dependence on the choice of coordinates. At the same time, the role of distinguished coordinates in physical applications is clarified. For example, these can be adapted to symmetry properties of the system.

(iv) The exterior calculus of differential forms is a very powerful method for practical calculations; one often finds the results faster than with older methods.

Space does not allow us always to give complete proofs and sufficient motivation. In these cases, we give detailed references to the literature (Refs. [1]–[8]) where these can be found. Many readers will have the requisite mathematical knowledge to skip this part after familiarizing themselves with our notation (which is quite standard). This is best done by looking at the collection of important formulae at the end (p. 70).

1. Differentiable Manifolds

A manifold is a topological space which locally looks like the space $\mathbb{R}^n$ with the usual topology.

Definition 1.1: An *n-dimensional topological manifold* M is a topological Hausdorff space with a countable base, which is locally homeomorphic to $\mathbb{R}^n$. This means that for every point $p \in M$ there is an open neighborhood U of p and a homeomorphism

$$h: U \to U'$$

which maps U onto an open set $U' \subset \mathbb{R}^n$.

As an aside, we note that a topological manifold M also has the following properties:

(i) M is σ-compact;

(ii) M is paracompact and the number of connected components is at most denumerable.

The second of these properties is particularly important for the theory of integration. For a proof, see e.g. [2], Chap. II, Sect. 15.

Definition 1.2.: If M is a topological manifold and $h: U \to U'$ is a homeomorphism which maps an open subset $U \subset M$ onto an open subset $U' \subset \mathbb{R}^n$, then h is a *chart* of M and U is called the *domain of the chart* or *local coordinate neighborhood*. The coodinates $(x^1, \ldots, x^n)$ of the image $h(p) \in \mathbb{R}^n$ of a point $p \in U$ are called the *coordinates* of p in the chart. A set of charts $\{h_\alpha \mid \alpha \in I\}$ with domains U_α is called an *atlas* of M, if $\bigcup_{\alpha \in I} U_\alpha = M$. If h_α and h_β are two charts, then both define homeomorphisms on the intersection of their domains $U_{\alpha\beta} := U_\alpha \cap U_\beta$; one thus obtains a homeomorphism $h_{\alpha\beta}$ between two open sets in $\mathbb{R}^n$ via the commutative diagram:

$$\begin{array}{ccccc} & & U_{\alpha\beta} & & \\ & {\scriptstyle h_\alpha}\swarrow & & \searrow{\scriptstyle h_\beta} & \\ U'_\alpha \supset h_\alpha(U_{\alpha\beta}) & & \xrightarrow{\quad h_{\alpha\beta}\quad} & & h_\beta(U_{\alpha\beta}) \subset U'_\beta . \end{array}$$

Thus $h_{\alpha\beta} = h_\beta \circ h_\alpha^{-1}$ on the domain where the mapping is defined (the reader should draw a figure). This mapping gives a relation between the coordinates in the two charts and is called a *change of coordinates*, or *coordinate transformation*. Sometimes, particularly in the case of charts, it is useful to include the domain of a mapping in the notation; thus, we write (h, U) for the mapping $h\colon U \to U'$.

Definition 1.3: An atlas defined on a manifold is said to be *differentiable* if all of its coordinate changes are differentiable mappings. For simplicity, unless otherwise stated, we shall always mean differentiable mappings of class C^∞ on $\mathbb{R}^n$ (the derivatives of all orders exist and are continuous). Obviously, for all coordinate transformations one has (on the domains for which the mappings are defined):

$$h_{\alpha\alpha} = \text{identity and } h_{\beta\gamma} \circ h_{\alpha\beta} = h_{\alpha\gamma}, \text{ so that } h_{\alpha\beta}^{-1} = h_{\beta\alpha},$$

and hence the inverses of the coordinate transformations are also differentiable. They are thus diffeomorphisms.

If $\mathscr{A}$ is a differentiable atlas defined on a manifold M, then the atlas $\mathscr{D}(\mathscr{A})$ contains those charts for which all coordinate changes among charts from $\mathscr{A}$ are differentiable. The atlas $\mathscr{D}(\mathscr{A})$ is then also differentiable since, locally, a coordinate change $h_{\beta\gamma}$ in $\mathscr{D}(\mathscr{A})$ can be written as a composition $h_{\beta\gamma} = h_{\alpha\gamma} \circ h_{\beta\alpha}$ of two other coordinate changes with a chart $h_\alpha \in \mathscr{A}$ and differentiability is a local property. The atlas $\mathscr{D}(\mathscr{A})$ is clearly maximal. In other words, $\mathscr{D}(\mathscr{A})$ cannot be enlarged by the addition of further charts, and is the largest atlas which contains $\mathscr{A}$. Thus, every differentiable atlas $\mathscr{A}$ determines uniquely a maximal differentiable atlas $\mathscr{D}(\mathscr{A})$ such that $\mathscr{A} \subset \mathscr{D}(\mathscr{A})$. Furthermore, $\mathscr{D}(\mathscr{A}) = \mathscr{D}(\mathscr{B})$ if and only if the atlas $\mathscr{A} \cup \mathscr{B}$ is differentiable.

Definition 1.4: A *differentiable structure* on a topological manifold is a maximal differentiable atlas. A *differentiable manifold* is a topological manifold, together with a differentiable structure.

In order to define a differentiable structure on a manifold, one must specify a differentiable atlas. In general, one specifies as small an atlas as possible, rather than a maximal one, which is then obtained as described above. We shall tacitly assume that all the charts and atlases of a manifold having a differentiable structure $\mathscr{D}$ are contained in $\mathscr{D}$. As a shorthand notation, we write M, rather than $(M, \mathscr{D})$ to denote a differentiable manifold.

Examples: (a) $M = \mathbb{R}^n$. The atlas is formed by the single chart: $(\mathbb{R}^n, \text{identity})$. (b) Any open subset of a differentiable manifold has an obvious differentiable structure. It may have others as well.

Definition 1.5: A continuous mapping $\varphi\colon M \to N$ from one differentiable manifold to another is said to be *differentiable at the point*