# Geometry of Banach spaces

## selected topics

Joseph Diestel.

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann



485

Joseph Diestel

# Geometry of Banach Spaces – Selected Topics

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## Preface

These notes were the subject of lectures given at Kent State University during the 1973-74 academic year. At that time, it was already clear that the geometry of Banach spaces (in the form of convexity and smoothness type considerations) would play a central role in the theory of Radon-Nikodym differentiation for vector-valued measures. This was the object of the course: to acquaint my students (and, to a large extent, myself) with the geometry of Banach spaces. Naturally, the logical finish to the courses was a discussion of the Radon-Nikodým theorem viewed from a purely geometric perspective.

Some words about the organization of the notes.

The first chapter deals with the plenitude of support functionals to closed bounded convex subsets of a Banach space. Two results are focal: the Bishop-Phelps subreflexivity theorem and James' characterization of weak compactness. I feel that these are among the deepest results of modern functional analysis and have tried throughout the notes to apply them whenever possible. When the Deity allowed for theorems like these to be proved, He meant for them to be used! This chapter is closed with an application to operators attaining their norm which uses the theory to topological tensor products for its proof; this is the only excursion concerning prerequisites outside of elementary functional analysis and is a one-time affair. The principle purpose here is to highlight the severe restriction placed upon a Banach space (or pair of Banach spaces) that every operator attain its norm; it also is an interesting application of James' theorem.

Chapter Two deals with the basics of convexity and smoothness. It provides an excellent collection of applications of both principles set forth in the first chapter. This chapter's topics are classical (except for the use of the aforementioned principles) with the possible exception of the last section on normal structure.

Chapter Three contains some of the most beautiful results in all these notes; the theory of series in uniformly convexifiable spaces is developed. Much here is left unsaid, of course, but an effort has been made to present the basic results. One truly major omission is the theory of superreflexive Banach spaces; actually, the exclusion of a discussion of superreflexivity is due to the simple fact that I did not have time to discuss it in the lectures.

The first discussion of what might be called the isomorphic theory of the geometry of Banach spaces is contained in Chapter Four where the classical renorming theorems are presented. The theory of renorming goes back to the days of Clarkson though real applications of the theory awaited the development of infinite dimensional topology. Nonetheless, the pioneers of the area proved some beautiful (and useful) results on changing norms.

The theory of weakly compactly generated Banach spaces is taken up in Chapter Five. Again, these lectures were aimed at students of vector-valued measures; a fundamental fact-of-life in vector measures is that every vector measure has its range in a weakly compactly generated Banach space. Of course, the hand of Lindenstrauss is heavily felt in this chapter (as it is throughout these notes). I have tried to give the central results of weakly compactly generated Banach spaces as they were at the time of the course. There have been a few developments since the course but the notes on this section are fairly up to date.

As remarked before, Chapter Six is concerned with the Radon-Nikodým theorem for vector valued measures. Most of the material of this chapter is quite recent (the same could be said for Chapter Five). Our presentation is geometric and founded on the Davis-Huff-Maynard-Phelps-Rieffel characterization of spaces with the Radon-Nikodým property. We have touched only on the geometric aspects of the Radon-Nikodým theorem; a much more comprehensive discussion (albeit from a different point of view) is contained in the forthcoming monograph, "Vector Measures" by J. J. Uhl, Jr. and myself.

As is to be expected, I have benefitted from discussions with a number of mathematicians. Particularly great is my debt of gratitude to Professors Bill Davis, Tadeusz Figiel and Bill Johnson of Ohio State University, Professors Bob Huff and Peter Morris of Pennsylvania State University, Professor Bob Phelps of the University of Washington and Professors Johnnie Baker and Bob Lohman of Kent State University. Much of my understanding of the material in these notes came from reflecting on the results contained herein in light of discussions with these people. They have provided me with preprints of their work as well as elegant proofs of several, previously cumbersome, theorems.

I also owe a great deal to the students who sat through the class in which this material was presented. They cleared up many inaccuracies and forced me to clarify some very muddy arguments. Particularly, I extend thanks to Dr. Barbara Faires and Mr. Terry Morrison.

Thanks also go to Mr. J. Tischer for careful reading of much of the finished manuscript and for a number of elegant arguments, now incorporated in the notes, replacing some rather cumbersome constructions.

Finally, sincerest thanks go to the most crucial link in the preparation of these notes: the Kent State University Mathematics Department secretaries, Julia Froble and Darlene May. Their patience with my poor handwriting and frequent cajoling was unbelievable.

THROUGHOUT THESE NOTES ALL OUR BANACH SPACES ARE ASSUMED TO BE REAL BANACH SPACES. The reader will note that many of the proofs hold with minor modifications for the complex case as well. However, several proofs require rather drastic surgery to be adapted to the complex case; rather than take a chance with "a successful operation in which the patient died", I have presented only the proofs for real scalars feeling that here is where the intuition best serves valid understanding of the geometric phenomena discussed.

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## CHAPTER ONE

## SUPPORT FUNCTIONALS FOR CLOSED BOUNDED CONVEX SUBSETS OF A BANACH SPACE

This chapter contains two very deep results concerning support functionals for closed bounded convex subsets of a Banach space: the Bishop-Phelps subreflexivity theorem and James' characterization of weakly compact sets. They will form the cornerstone for the first part of these notes.

Let us establish some terminology.

Let X be a Banach space.

A convex set $K \subseteq X$ is a <u>convex cone</u> if it is closed under positive scalar multiples. Let $C \subseteq X$ and $x_0 \in C$; the convex cone K <u>supports C at $x_0$</u> if $K + x_0 \cap C = \{x_0\}$.

Note that if K is a convex cone with nonempty interior $K^0$ and C is convex with K supporting C at $x_0$, then by the Hahn-Banach theorem (separation form) there exists $g \in X^*$ such that $\sup g(C) \le \inf g(K + x_0)$. Since $x_0 \in C \cap K + x_0$, $g(x_0) = \sup g(C)$.

So let $f \in X^*$ with $\|f\| = 1$ and let $k \ge 0$. Define

$$K(f, k) = \{x \in X: \|x\| \le kf(x)\};$$

K(f, k) is a closed convex cone.

If $k > 1$ then there exists $x_f \in X$ with $\|x_f\| = 1$ such that $1/k < f(x_f)$. By continuity of f and $\| \ \|$ it follows that the interior of K(f, k) is nonempty.

<u>Lemma 1</u>: Suppose C is closed and convex; $f \in X^*$ with $\|f\| = 1$; suppose f is bounded on C and let $k > 0$.

If $z \in C$, then there exists $x_0 \in C$ such that $x_0 - z \in K(f, k)$ and K(f, k) supports C at $x_0$.

Proof:

Partially order C as follows:

$x, y \in C$ satisfy $x \leq y$ iff $x - y \in K(f, k)$

iff $\|x - y\| \leq k\,f(x - y) = k\,f(x) - k\,f(y)$.

We will show $(C, \leq)$ has a maximal element $x_0 \geq z$. Let $\mathfrak{Z}$ = collection of $x \geq z$ and let W be a chain in $\mathfrak{Z}$; note that $\{f(w): w \in E\}$ is a bounded, monotone real net hence converges to its least upper bound. In particular, it is Cauchy, so W is a Cahucy set in the norm topology and hence convergent to some $x_w \in C$. Continuity of f and $\|\cdot\|$ now imply $x_w \in \mathfrak{Z}$ and $x_w$ is an upper bound for W.

Thus every chain has an upper bound and by Zorn's Lemma, $\mathfrak{Z}$ has a maximal element $x_0$ which is clearly $\geq z$.

To show K(f, k) supports C at $x_0$, note that $x_0 \in C \cap [K(f, k) + x_0]$. In fact, $x_0$ is the only point in this intersection. To see this, suppose $\bar{x} \in C \cap [K(f, k) + x_0]$. Then $\bar{x} - x_0 \in K(f, k)$. As $\bar{x} \in C$, $x_0 \leq \bar{x}$. On the other hand, $z \leq x_0$ and $x_0 \leq \bar{x}$ so $z \leq \bar{x}$ for each $z \in C$. Thus $\bar{x} \leq x_0$ and $\bar{x} = x_0$ as was claimed.

Lemma 2: Suppose $\epsilon > 0$, $\|f\| = \|g\| = 1$; and $(\|x\| \leq 1,\ f(x) = 0 \Rightarrow |g(x)| \leq \epsilon/2)$ then either $\|f-g\|$ or $\|f+g\| \leq \epsilon$.

Proof:

Let h be such that h coincides with g on the kernel of f and $\|h\| \leq \epsilon/2$ (Hahn-Banach). Then $g-h = 0$ on kernel of f so $g-h = \alpha f$ for some $\alpha$. Note that

$$|1 - |\alpha|| = |\|g\| - \|g-h\|| \leq \|h\| \leq \epsilon/2.$$

Thus if $\alpha \geq 0$ then

$$\|f-g\| = \|(1-\alpha)f-h\| \leq |1-\alpha| + \|h\| \leq \epsilon.$$



If $\alpha < 0$ then

$$\|f+g\| = \|(1+\alpha)f+h\| \leq |1+\alpha| + \|h\| \leq \epsilon.$$

This completes the proof.

Lemma 3: If $0 < \epsilon < 1$ and $\|f\| = \|g\| = 1$ and $k > 1 + 2/\epsilon$, and g is non-negative on K(f, k), then $\|f-g\| \leq \epsilon$.

Proof:

Let $x \in X$ be such that $\|x\| = 1$, $f(x) > (1 + 2/\epsilon)/k$. If $y \in X$ has $\|y\| < 2/\epsilon$, and $f(y) = 0$, then $\|x \pm y\| \leq 1 + 2/\epsilon < kf(x) = kf(x \pm y)$ so $x \pm y \in K(f, k)$. Thus $g(x \pm y) \geq 0$ and so $|g(y)| \leq g(x) \leq \|x\| = 1$. Now $\|f+g\| \leq \epsilon$ or $\|f-g\| \leq \epsilon$, by Lemma 2.

Claim: $\|f+g\| \leq \epsilon$ is impossible. Note $\|f\| = 1$ so as $1/k$, $\epsilon < 1$, there is $z \in X$ with $\|z\| = 1$ and with $f(z) > \max(1/k, \epsilon)$. But then $f(z) > (1/k)\|z\|$ so $z \in K(f, k)$ so that $g(z) \geq 0$ and we have

$$\|f+g\| \geq (f+g)(z) > \epsilon.$$

It follows that $\|f-g\| \leq \epsilon$. ■

Theorem (Bishop-Phelps): Let C be a closed bounded convex set in the Banach space X, then the collection of functionals that achieve their maximum on C is dense in $X^*$.

Proof:

We can assume $0 \in C$ and need only approximate those $f \in X^*$ with $\|f\| = 1$. Let $0 < \epsilon < 1$ be given and choose $k > 1 + 2/\epsilon > 1$. Then K(f, k) is a closed convex cone with nonempty interior. Now apply Lemma 1 to C with $z = 0$ yielding $x_0 \in C$ with $x_0 \in K(f, k)$ and $(K(f, k) + x_0) \cap C = \{x_0\}$ so K(f, k) supports C at $x_0$. Now there is $g \in X^*$ $(g \neq 0)$ with $\|g\| = 1$ such that

$$\sup g(C) = g(x_0) = \inf g(K(f, k) + x_0)$$
$$= \inf g(K(f, k)) + g(x_0).$$

In particular, $\inf g(K(f, k)) = 0$ so $g \geq 0$ on $K(f, k)$ and lemma 3 implies $\|f - g\| \leq \epsilon$.

Our next results are due to R. C. James who has kindly furnished the mathematical public with relatively easy proofs of these beautiful and deep facts.

We start with

Lemma 4: Let $0 < \theta < 1$ and let $(f_n) \subseteq X^*$ with $\|f_n\| = 1$. Now suppose $\|f\| \geq \theta$ for $f \in co\{f_n\}$. If $\lambda_n > 0$ and $\Sigma_n \lambda_n = 1$ then there is $\alpha$ with $\theta \leq \alpha \leq 1$ and a sequence $(g_n)$ such that

(i) $g_n \in co\{f_n, f_{n+1}, \cdots\}$

(ii) $\|\Sigma_n \lambda_n g_n\| = \alpha$

(iii) for each n, $\|\sum_1^n \lambda_i g_i\| < \alpha \left[1 - \theta \sum_{n+1}^{\infty} \lambda_i\right]$.

Proof:

Choose $(\epsilon_m) > 0$ such that

(1) $$\sum_n \frac{\lambda_n \epsilon_n}{(\sum_{i=n+1}^{\infty} \lambda_i)(\sum_{i=n}^{\infty} \lambda_i)} < 1 - \theta.$$

We manufacture the sequence $(g_n)$ as follows.

Let

$$\alpha_1 = \inf\{\|g\|: g \in co\ (f_n)\};$$

clearly $\theta \leq \alpha_1 \leq 1$. Choose $g_1 \in co(f_n)$ such that $\|g_1\| \leq \alpha_1(1 + \epsilon_1)$.

Let

$$\alpha_2 = \inf\{\|\lambda_1 g_1 + (\sum_{i=2}^{\infty} \lambda_i) g\|: g \in co(f_2, f_3, \cdots)\};$$

clearly $\theta \leq \alpha_1 \leq \alpha_2 \leq 1$. Choose $g_2 \in co(f_2, f_3, \cdots)$ such that $\|\lambda_1 g_1 + (\sum_2^{\infty} \lambda_i) g_2\| < \alpha_2\ (1 + \epsilon_2)$.



Continuing in this fashion we obtain $(g_n)$ so that $g_n \in co(f_n, f_{n+1}, \cdots)$, $(\alpha_n)$ so that $\theta \leq \alpha_n \leq \alpha_{n+1} \leq 1$ and

(2) $$\|\sum_{i=1}^{n-1} \lambda_i g_i + (\sum_n^{\infty} \lambda_i) g_n\| < \alpha_n (1 + \epsilon_n),$$

where $\alpha_n = \inf\{\|\sum_{i=1}^{n-1} \lambda_i g_i + (\sum_n^{\infty} \lambda_i) g\|: g \in co(f_n, f_{n+1}, \cdots)\}$.

We have now that $\theta \leq \alpha_n \uparrow \alpha = \|\Sigma_n \lambda_n g_n\| \leq 1$. So $(g_n)$ satisfies (i) and (ii). It remains to show that $(g_n)$ satisfies (iii).

We first establish that

(3) $$\|\sum_1^m \lambda_i g_i\| < \left[\sum_{m+1}^{\infty} \lambda_i\right] \left[\frac{\lambda_m \alpha_m (1 + \epsilon_m)}{(\sum_m^{\infty} \lambda_i)(\sum_{m+1}^{\infty} \lambda_i)} + \frac{1}{(\sum_m^{\infty} \lambda_i)} \|\sum_1^{m-1} \lambda_i g_i\|\right]$$

holds for all n.

$$\|\sum_{i=1}^{n} \lambda_i g_i\| = \|\frac{\sum_{i=n}^{\infty} \lambda_i}{\sum_{i=n}^{\infty} \lambda_i} \sum_{i=1}^{n-1} \lambda_i g_i + \frac{\sum_{i=n}^{\infty} \lambda_i}{\sum_{i=n}^{\infty} \lambda_i} \lambda_n g_n\|$$

$$= \|\frac{\lambda_n + \sum_{i=n+1}^{\infty} \lambda_i}{\sum_{i=n}^{\infty} \lambda_i} \sum_{i=1}^{n-1} \lambda_i g_i + \frac{\lambda_n \sum_{i=n}^{\infty} \lambda_i}{\sum_{i=n}^{\infty} \lambda_i} g_n\|$$

$$\leq \frac{\lambda_n}{\sum_{i=n}^{\infty} \lambda_i} \|\sum_{i=1}^{n-1} \lambda_i g_i + (\sum_{i=n}^{\infty} \lambda_i) g_n\| + \frac{\sum_{i=n+1}^{\infty} \lambda_i}{\sum_{i=n}^{\infty} \lambda_i} \|\sum_{i=1}^{n-1} \lambda_i g_i\|$$

(by (2)) $$< \frac{\lambda_n}{\sum_{i=n}^{\infty} \lambda_i} (\alpha_n (1 + \epsilon_n)) + \frac{\sum_{i=n+1}^{\infty} \lambda_i}{\sum_{i=n}^{\infty} \lambda_i} \|\sum_{i=1}^{n-1} \lambda_i g_i\|$$

$$= (\sum_{i=n+1}^{\infty} \lambda_i)\left(\frac{\lambda_n \alpha_n (1 + \epsilon_n)}{\sum_{i=n}^{\infty} \lambda_i \sum_{i=n+1}^{\infty} \lambda_i} + \frac{1}{\sum_{i=n}^{\infty} \lambda_i} \|\sum_{i=1}^{n-1} \lambda_i g_i\|\right)$$

and we have (3) holding for all n.

Now apply equation (3) to $\|\sum_1^{n-1}\lambda_i g_i\|$ with n = n - 1 so

$$\|\sum_{i=1}^{n-1}\lambda_i g_i\| < (\sum_{i=n}^{\infty}\lambda_i)\left(\frac{\lambda_{n-1}\alpha_{n-1}(1+\epsilon_{n-1})}{(\sum_{i=n-1}^{\infty}\lambda_i)(\sum_{i=n}^{\infty}\lambda_i)} + \frac{1}{\sum_{i=n-1}^{\infty}\lambda_i}\|\sum_{i=1}^{n-2}\lambda_i g_i\|\right)$$

yields

$$\|\sum_{i=1}^{n}\lambda_i g_i\| < (\sum_{i=n+1}^{\infty}\lambda_i)\left(\frac{\lambda_n\alpha_n(1+\epsilon_n)}{(\sum_{i=n}^{\infty}\lambda_i)(\sum_{i=n+1}^{\infty}\lambda_i)} + \frac{\lambda_{n-1}\alpha_{n-1}(1+\epsilon_{n-1})}{(\sum_{i=n-1}^{\infty}\lambda_i)(\sum_{i=n}^{\infty}\lambda_i)} + \frac{1}{\sum_{i=n-1}^{\infty}\lambda_i}\|\sum_{i=1}^{n-2}\lambda_i g_i\|\right).$$

Now iterate again using n = n - 2 and get finally

$$\|\sum_{i=1}^{n}\lambda_i g_i\| < (\sum_{i=n+1}^{\infty}\lambda_i)\sum_{k=1}^{n}\frac{\lambda_k\alpha_k(1+\epsilon_k)}{(\sum_{i=k}^{\infty}\lambda_i)(\sum_{i=k+1}^{\infty}\lambda_i)} \qquad \text{(by definition of } \alpha)$$

$$\le \alpha(\sum_{i=n+1}^{\infty}\lambda_i)\sum_{k=1}^{n}\frac{\lambda_k(1+\epsilon_k)}{(\sum_{i=k}^{\infty}\lambda_i)(\sum_{i=k+1}^{\infty}\lambda_i)} \qquad \text{(by (1))}$$

$$\le \alpha(\sum_{i=n+1}^{\infty}\lambda_i)\left(\sum_{k=1}^{n}\frac{\lambda_k}{(\sum_{i=k}^{\infty}\lambda_i)(\sum_{i=k+1}^{\infty}\lambda_i)} + (1-\theta)\right)$$

$$= \alpha(\sum_{i=n+1}^{\infty}\lambda_i)\left(\sum_{k=1}^{n}\left(\frac{1}{\sum_{i=k+1}^{\infty}\lambda_i} - \frac{1}{\sum_{i=k}^{\infty}\lambda_i}\right) + (1-\theta)\right)$$

$$\le \alpha(\sum_{i=n+1}^{\infty}\lambda_i)\left(\frac{1}{\sum_{i=n+1}^{\infty}\lambda_i} - \frac{1}{\sum_n\lambda_n} + 1 - \theta\right) \qquad \text{(since } \sum_n\lambda_n = 1)$$

$$= \alpha(\sum_{i=n+1}^{\infty}\lambda_i)\left(\frac{1}{\sum_{i=n+1}^{\infty}\lambda_i} - \theta\right)$$

$$= \alpha(1 - \theta(\sum_{i=n+1}^{\infty}\lambda_i)).$$

<u>Theorem 2 (James-Klee)</u>: Let X be a <u>separable</u> Banach space. TFAE:

(i) X is not reflexive

(ii) if $0 < \theta < 1$, then there exists $(f_n) \subseteq X^*$ such that $\|f_n\| \le 1$, $\|f\| \ge \theta$ whenever $f \in co(f_n)$ and $f_n \to 0$ weak star.

(iii) if $0 < \theta < 1$ and $\lambda_n > 0$ and $\Sigma_n \lambda_n = 1$, then there exists $\alpha$: $\theta \le \alpha \le 1$ and there exists $(g_n) \subseteq X^*$ such that $g_n \to 0$ weak-star, $\|\Sigma_n \lambda_n g_n\| = \alpha$ and for each n $\|\Sigma_1^n \lambda_i g_i\| \le \alpha(1 - \theta\,\Sigma_{n+1}^{\infty}\lambda_i)$;

(iv) there exists $f \in X^*$ not achieving its norm.

<u>Proof</u>:

Let X be nonreflexive and choose $F \in X^{**}$ such that $\|F\| < 1$ and $d(F, X) > \theta$. Let $(x_n)$ be dense in X and recall <u>Helly's theorem:</u>

[$h_1^*, \cdots, h_n^* \in Z^*$; $c_1, \cdots, c_n \in$ scalars, then for there to exist for each $\epsilon > 0$, $z \in Z$ such that $\|z\| \le M + \epsilon$ and $h_i^* z = c_i$ it is necessary and sufficient that given scalars $a_1, \cdots, a_n$, $|\sum_1^n a_i c_i| \le M\|\sum_1^n a_i h_i\|$

So let $h_i^* = x_1$, $h_2^* = x_2$, $\cdots$, $h_{n-1}^* = x_{n-1}$, $h_n^* = F$

$Z = X^*$, $c_1 = c_2 = \cdots = c_{n=1} = 0$, $c_n = \theta$ ($z = f \in X^*$).. Then we have:

$$|\sum_{i=1}^{n} a_i c_i| = |a_n|\,|c_n| = |a_n|\theta = |a_n|\frac{\theta}{\text{dist}(F,X)}\,\text{dist}(F,X)$$

$$\le |a_n|\frac{\theta}{\text{dist}(F,X)}\|F + \sum_{i=1}^{n-1}\frac{a_i}{|a_n|}x_i\|$$

$$= \frac{\theta}{\text{dist}(F,X)}\|a_n F + \sum_{i=1}^{n-1} a_i x_i\|.$$

There is $f_{n-1} \in X^*$ such that $\|f_{n-1}\| < 1$ and such that $f_{n-1}(x_i) = 0$ $1 \le i \le n-1$ and $F(f_{n-1}) = \theta$. Now $(f_n)$ is such that $f_n(x) \to 0$ for $x \in X$

and if $f \in co(f_n)$, $f = \sum_1^k \beta_i f_i$ so $F(f) = \sum_1^k \beta_i F(f_i) = \theta \Rightarrow \|f\| \geq \theta$. Thus (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) follows easily from Lemma 4.

(iii) $\Rightarrow$ (iv):

Claim: Given everything as in (iii) then $\Sigma_n \lambda_n g_n \in X^*$ does not attain its norm. In fact, let $x \in X$, $\|x\| \leq 1$. Then $g_n(x) \to 0$. Choose n such that $m \geq n+1 \Rightarrow g_m(x) < \alpha\theta$. Then

$$\begin{aligned}\Sigma_n \lambda_n g_n(x) &< \sum_{i=1}^n \lambda_i g_i(x) + \alpha\,\theta \sum_{i=n+1}^{\infty} \lambda_i \\ &\leq \left\|\sum_{i=1}^n \lambda_i g_i\right\| + \alpha\,\theta \sum_{i=n+1}^{\infty} \lambda_i \\ &< \alpha\Big(1 - \theta \sum_{i=n+1}^{\infty} \lambda_i\Big) + \alpha\,\theta \sum_{i=n+1}^{\infty} \lambda_i = \alpha.\end{aligned}$$

(iv) $\Rightarrow$ (i). Suppose X is reflexive and $\varphi \in X^*$, then there is $F \in X^{**}$ such that $\|F\| = 1$ and $\|\varphi\| = F(\varphi)$. But X reflexive $\Rightarrow$ there is $x \in X$, $\|x\| = 1$ such that $g(x) = x(g) = F(g)$ for $g \in X^*$, so $\varphi(x) = F(\varphi) = \|\varphi\|$ and each $f \in X^*$ attains its norm.

Our next task is to extend the above result to the non-separable case. For this a bit more work is required. We start with some

Notation: Let $(\varphi_n)$ be bounded in $X^*$, (X a Banach space). Denote by $L(\varphi_n)$ those $w \in X^*$ such that

$$\underline{\lim}\ \varphi_n(x) \leq w(x) \leq \overline{\lim}\ \varphi_n(x) \qquad \text{for } x \in X.$$

Remark: $L(\varphi_n) \neq \emptyset$. Consider the map $X \to \ell_\infty$ given by $x \to (\varphi_n(x))$. Let $\ell \in \ell_\infty^*$ be a Banach limit, then $\underline{\lim}\ \varphi_n(x) \leq \ell(x) \leq \overline{\lim}\ \varphi_n(x)$

Lemma 5: Let X be any Banach space, $0 < \theta < 1$, $(f_n) \subset X^*$ $\|f_n\| \leq 1$. Suppose that $\|f - w\| \geq \theta$ for any $f \in co(f_n)$ and any $w \in L(f_n)$ and that $\lambda_n > 0$ with $\Sigma_n \lambda_n = 1$. Then there exists $\alpha$ such that $\theta \leq \alpha \leq 2$ and $(g_n) \subset X^*$ such that $\|g_n\| \leq 1$ with the following properties



- for each $w \in L(g_n)$, $\|\Sigma_n \lambda_n(g_n - w)\| = \alpha$

and

- for each $w \in L(g_n)$ and each n,

$$\left\|\sum_{i=1}^n \lambda_i(g_i - w)\right\| < \alpha\Big(1 - \theta\Big(\sum_{n+1}^{\infty} \lambda_i\Big)\Big).$$

Proof:

Choose $\epsilon_n > 0$ such that

$$\text{(1)} \qquad \Sigma_n \frac{\lambda_n \epsilon_n}{\left(\sum_{n+1}^{\infty} \lambda_i\right)\left(\sum_n^{\infty} \lambda_i\right)} < 1 - \theta.$$

Let $(\psi_i^{(0)}) = (f_i)$. Define

$$\alpha_1 = \inf\{\sup\{\|g - w\| : w \in L(\varphi_i)\}\},$$

where the infimum is taken over those $g \in co(f_n)$ and those sequences $(\varphi_i)$ satisfying $\varphi_k \in co(f_k, f_{k+1}, \cdots)$ for each k. Choose $g_1 \in co(f_n)$ and a sequence $(\varphi_i^{(1)})$ such that for each k $\varphi_k^{(1)} \in co(f_k, f_{k+1}, \cdots)$ with

$$\alpha_1 \leq \sup\{\|g_1 - w\| : w \in L(\varphi_i^{(1)})\} < \alpha_1(1 + \epsilon_1).$$

Now choose $w' \in L(\varphi_i^{(1)})$ so that

$$\alpha_1(1 - \epsilon_1) < \|g_1 - w'\| < \alpha_1(1 + \epsilon_1).$$

Let $\bar{x} \in X$, $\|\bar{x}\| \leq 1$ be such that

$$\alpha_1(1 - \epsilon_1) < (g_1 - w')(\bar{x}).$$

Then as

$$\underline{\lim}\ \varphi_i^{(1)}(\bar{x}) \leq w'(\bar{x})$$

there is a subsequence $(\psi_i^{(1)})$ of $(\varphi_i^{(1)})$ such that for any $w \in L(\psi_i^{(1)})$

we have

$$\underline{\lim}_i \varphi_i^{(1)}(\bar{x}) = \lim_i \psi_i^{(1)}(\bar{x}) = w(\bar{x}) \le w'(\bar{x}).$$

This means that we can replace w' by w above to obtain

$$\alpha_1(1 - \epsilon_1) < (g_1 - w)(\bar{x}).$$

Now let

$$\alpha_2 = \inf \{\sup\{\|\lambda_1 g_1 + (\sum_2^\infty \lambda_i) g - w\| : w \in L(\varphi_i)\}\}$$

where the infimum is taken over all $g \in co(\psi_2^{(1)}, \psi_3^{(1)}, \cdots)$ and all sequences $(\varphi_i)$ such that $\varphi_k \in co(\psi_k^{(1)}, \psi_{k+1}^{(1)}, \cdots)$. Choose $g_2 \in co(\psi_2^{(1)}, \psi_3^{(1)}, \cdots)$ and $(\varphi_i^{(2)})$ such that $\varphi_k^{(2)} \in co(\psi_k^{(1)}, \psi_{k+1}^{(1)}, \cdots)$ for all k and

$$\alpha_2 \le \sup \{\|\lambda_1 g_1 + (\sum_2^\infty \lambda_i) g_2 - w\| : w \in L(\varphi_i^2)\} < \alpha_2(1 + \epsilon_2).$$

Next choose $w' \in L(\varphi_i^{(2)})$ such that

$$\alpha_2(1 - \epsilon_2) < \|\lambda_1 g_1 + (\sum_2^\infty \lambda_i) g_2 - w'\| < \alpha_2 (1 + \epsilon_2)$$

and let $\bar{x} \in X$, $\|\bar{x}\| \le 1$ be such that

$$\alpha_2(1 - \epsilon_2) < \lambda_1 g_1(\bar{x}) + \sum_2^\infty \lambda_i g(\bar{x}) - w'(\bar{x}).$$

As above select a subsequence $(\psi_i^{(2)})$ of $(\varphi_i^{(2)})$ such that for $w \in L(\psi_i^{(2)})$ we have

$$\underline{\lim}_i \varphi_i^{(2)}(\bar{x}) = \lim_i \psi_i^{(2)}(\bar{x}) = w(\bar{x}) \le w'(\bar{x})$$

and again we can replace w' by w above.

Continuing in this fashion we let

(2) $$\alpha_n = \inf \{\sup\{\| \sum_{i=1}^{n-1} \lambda_i g_i + (\sum_n^\infty \lambda_i) g - w\| : w \in L(\varphi_i)\}\},$$

where the infimum is taken over those $g \in co(\psi_n^{(n-1)}, \psi_{n+1}^{(n-1)}, \cdots)$ and those sequences $(\varphi_i)$ satisfying $\varphi_k \in co(\psi_k^{(n-1)}, \psi_{k+1}^{(n-1)}, \cdots)$ for all k. Choose $g_n \in co(\psi_n^{(n-1)}, \psi_{n+1}^{(n-1)}, \cdots)$ and a sequence $(\varphi_i^{(n)})$ such that $\varphi_k^{(n)} \in co(\psi_k^{(n-1)}, \psi_{k+1}^{(n-1)}, \cdots)$ with

(3) $$\alpha_n \le \sup \{\| \sum_{i=1}^{n-1} \lambda_i g_i + (\sum_n^\infty \lambda_i) g_n - w\| : w \in L(\varphi_i^{(n)})\} < \alpha_n(1 + \epsilon_n).$$

Then choose $w' \in L(\varphi_i^{(n)})$ so that

$$\alpha_n(1 - \epsilon_n) < \| \sum_{i=1}^{n-1} \lambda_i g_i + (\sum_n^\infty \lambda_i) g_n - w'\| < \alpha_n(1 + \epsilon_n).$$

Let $\bar{x} \in X$, $\|\bar{x}\| \le 1$ be selected so that

(4) $$\alpha_n(1 - \epsilon_n) < \sum_{i=1}^{n-1} \lambda_i g_i(\bar{x}) + (\sum_n^\infty \lambda_i) g_n(\bar{x}) - w(\bar{x}).$$

Using

$$\underline{\lim}_i \varphi_i^{(n)}(\bar{x}) \le w'(\bar{x})$$

obtain a subsequence $(\psi_i^{(n)})$ of $(\varphi_i^{(n)})$ such that for each $w \in L(\psi_i^{(n)})$ we get

$$\underline{\lim}_i \varphi_i^{(n)}(\bar{x}) = \lim_i \psi_i^{(n)}(\bar{x}) = w(\bar{x}) \le w'(\bar{x}).$$

This allows us to replace w' in (4) by w.

This completes the construction.

By the fact that $L(g_n) \subseteq L(\varphi_i^{(n)})$ for each i, (3) and the replacement of w' by w in (4) we get for each n that

(5) $$\alpha_n(1 - \epsilon_n) < \| \sum_{i=1}^{n-1} \lambda_i g_i + (\sum_n^\infty \lambda_i) g_n - w\| < \alpha_n(1 + \epsilon_n)$$

for any $w \in L(g_n)$.

Note that $\|g_n\| \le 1$ for all n. Thus $g \in co(g_n)$ implies $\|g\| \le 1$. As

$\|w\| \le 1$ for $w \in L(g_n)$ we have $\alpha_n \le 2$. From (2) one easily derives that $\alpha_n \uparrow$. Of course, $\alpha_n \ge \theta$ by construction and the hypotheses of the lemma. We have then that $\alpha = \lim_n \alpha_n$ exists and satisfies (look at (5)),

$$\theta \le \alpha = \| \sum_n^1 \lambda_n (g_n - w)\| \le 2.$$

Thus we only have the estimate on $\| \sum_{i=1}^{n} \lambda_i (g_i - w)\|$ to establish. But the establishment of this is obtained directly as in Lemma 4 substituting $g_i - w$ for $g_i$ throughout. Finis.

<u>Theorem 3 (James)</u>: Let X be a Banach space. TFAE:

(i) X is non-reflexive;

(ii) if $0 < \theta < 1$ then there is a sequence $(f_n) \subset X^*$, $\|f_n\| \le 1$ and a subspace $X_0$ of X such that $\|f - w\| \ge \theta$ for all $f \in co(f_n)$ and for any $w \in X_0^\perp$ and such that $f_n(x) \to 0$ for each $x \in X_0$;

(iii) if $0 < \theta < 1$ and $\lambda_n \ge 0$ with $\sum_n^1 \lambda_n = 1$ then there exists an $\alpha$, $\theta \le \alpha \le 2$ and $(g_n) \subset X^*$ with $\|g_n\| \le 1$ such that for $w \in L(g_n)$ we have

$$\|\sum_n^1 \lambda_n (g_n - w)\| = \alpha$$

and

$$\| \sum_{i=1}^{n} \lambda_i (g_i - w)\| < \alpha\ (1 - \theta(\sum_{n+1}^{\infty} \lambda_i)).$$

(iv) there exists $f \in X^*$ which does not achieve its norm.

<u>Proof</u>:

(i) implies (ii). Let $X_0$ be a non-reflexive separable subspace of X. Apply Theorem 2 to $X_0$ to obtain a sequence $(f_n) \subset X^*$ $\|f_n\| \le 1$ with $f_n(x)$ 0 for each $x \in X_0$ and such that $\|f\|_{X_0} \ge \theta$ for all $f \in co(f_n)$. Now, for $f \in co(f_n)$, $w \in X_0^\perp$

$$\|f - w\| \ge \|f - w\|_{X_0} = \|f\|_{X_0} \ge \theta.$$

(ii) implies (iii) is immediate from Lemma 5.



(iii) implies (iv). Let $(\lambda_n)$ be chosen such that $\lambda_n \ge 0$ and $\Sigma_n \lambda_n = 1$ and such that there is $\Delta > 0$ with $0 < \Delta < \theta^2/2$ and $\lambda_{n+1} < \Delta(\lambda_n)$, holding for all n. <u>Assertion</u>: $\Sigma_n \lambda_n (g_n - w)$ does not achieve its norm for any $w \in L(g_n)$. In fact, if $x \in X$ has $\|x\| \le 1$ then $\underline{\lim}\ g_i\ (x) \le w(x)$ for all $w \in L(g_n)$. As $\theta \le \alpha$ we must have n such that

$$(g_{n+1} - w)(x) < \theta^2 - 2\Delta \le 2\theta - 2\Delta.$$

But then

$$\begin{aligned}\Sigma_n \lambda_n (g_n - w)(x) &< \sum_{i=1}^{n} \lambda_n (g_n - w)(x) + (\alpha\theta - 2\Delta)\lambda_{n+1} \\ &\quad + \sum_{n+2}^{\infty} \lambda_n (g_n - w)(x) \\ &\le \|\sum_{i=1}^{n} \lambda_n (g_n - w)\| + (\alpha\theta - 2\Delta)\lambda_{n+1} \\ &\quad + 2 \sum_{n+2}^{\infty} \lambda_i \\ &\le \alpha(1 - \theta) \sum_{n+1}^{\infty} \lambda_i + (\alpha\theta - 2\Delta)\lambda_{n+1} + 2\Delta \sum_{n+1}^{\infty} \lambda_i \\ &= \alpha - (\alpha\theta - 2\Delta) \sum_{n+2}^{\infty} \lambda_i < \alpha.\end{aligned}$$

But $\alpha$ is the norm of $\Sigma_n \lambda_n (g_n - w)$. This completes the proof of this implication and since (iv) implies (i) is same as in Theorem 2 effectively finishes the proof of this theorem.

<u>Theorem 4</u>: Let B be a separable, bounded, weakly closed subset of the quasi-complete locally convex space X.

TFAE:

(i) B is not weakly compact;

(ii) there exists $\theta > 0$ and an equicontinuous sequence $(f_n)$ in $X^*$ such that $s_B(f) = \sup \{|f(b)| : b \in B\} \ge \theta$ for all $f \in co(f_n)$ and $f_n(b) \to 0$ for all $b \in B$;

(iii) there exists $\theta > 0$ such that if $\lambda_n > 0$ and $\Sigma_n \lambda_n = 1$ then for some $\alpha \geq \theta$ and some equicontinuous $(g_n) \subset X^*$ we have $g_n(b) \to 0$ for each $b \in B$ and $s_B(\Sigma_n \lambda_n g_n) = \alpha$ and $s_B(\sum_{i=1}^{\infty} \lambda_i g_i) < \alpha(1 - \theta(\sum_{n+1}^{\infty} \lambda_i))$ holding for all n;

(iv) there exists $g \in X^*$ which does not achieve its supremum on B.

Proof:

(i) implies (ii). Recall that if X is a locally convex space then there exists a family $\{X_\gamma : \gamma \in \Gamma\}$ of Banach spaces $X_\gamma$ such that X is a subspace of $\Pi_\Gamma X_\gamma$. X is quasi complete and B is bounded so $\overline{co}(B)$ is closed and bounded in X hence in $\Pi_\Gamma X_\gamma$. By Mazur's theorem $\overline{co}$ B is weakly closed in $\Pi_\Gamma X_\gamma$ and B is weakly closed in $\overline{co}$ B so B is weakly closed in $\Pi_\Gamma X_\gamma$. Note that $\Pi_\gamma B = \gamma$th projection of B is separable, bounded in $X_\gamma$. If B were not weakly compact then one of the $\Pi_\gamma$ B's would be non-weakly compact. Thus we have reduced our considerations to X being a Banach space and of course we can assume that the closed linear span of B is all of X so we are in situation of X being separable.

Consider the functional $s_B$ defined on $X^*$ by

$$s_B(f) = \sup\{|f(b)| : b \in B\}.$$

When we refer to $X^*$ in the $s_B$ topology we will call it C. Consider B in the weak topology $\sigma(X, X^*)$ and in the $\sigma(C^*, C)$-topology (B is a subset of the dual $C^*$ of the normed linear space C). We note that $(B, \sigma(X, X^*))$ and $(B, \sigma(C^*, C))$ coincide.

Consider $x \in B$ as a functional in $C^*$. Then

$$|x(f)| = |f(x)| \leq s_B(f)$$

so $\|x\|_{C^*} \leq 1$. Thus $B \subseteq$ closed unit ball of $C^*$. But the closed unit ball of $C^*$ is $\sigma(C^*, C)$ compact by Alaoglu's theorem. Thus if $X \cap C^* = C^*$, B would be weakly compact since it is a closed subset of the unit ball of



$C^*$ in the topology $\sigma(C^*, C)$ (which coincides with the $\sigma(X, X^*)$ topology on B). Hence as we've assumed B is not weakly compact it must be so that $X \cap C^* \subsetneq C^*$.

Thus there is an $\eta \in C^* \setminus X$. Let $M = \sup\{\|x\| : x \in B\}$, $\|\eta\|$ be the norm of $\eta$ in $X^{**}$ and $\|\eta\|_{C^*}$ be the norm of $\eta$ in $C^*$. Observe that $\|\eta\| \leq M\|\eta\|_{C^*}$. Now X is complete and is closed in $X^{**}$ so distance $(\eta, X) = 2\Delta > 0$. Let $(x_n)$ be dense in X, and select a sequence $(f_n) \subset X^*$ such that $\|f_n\| < 1$, $\eta f_n = \Delta$, $f_n(x_1) = f_n(x_2) = \cdots = f_n(x_n) = 0$ as in proof of (i) implies (ii) in Theorem 2. The sequence $(f_n)$ is uniformly bounded hence equicontinuous and clearly satisfies all else required of them including

$$s_B(f) \geq \theta = \frac{\Delta}{\|\eta\|_{C^*}}$$

for all $f \in co(f_n)$ since $\eta(f) = \Delta$ for all $f \in co(f_n)$.

The rest of the implications of the theorem are performed formally as in Theorem 1 and are not repeated herein.

Utilizing the fact that weakly compact and countably compact sets coincide in quasi-complete locally convex spaces to reduce the general case to the separable case, one can easily derive as well the following:

Theorem 5 (James): Let B be a weakly closed, bounded subset of the quasi-complete locally convex space X. TFAE:

(i) B is non weakly compact;

(ii) there exists a $\theta > 0$, a subset $B_0$ of B and an equicontinuous sequence $(f_n)$ of members of $X^*$ for which

$$s_B(f - w) \geq \theta$$

for all $f \in co(f_n)$ and $w \in B_0^{\perp}$ and $\lim_n f_n(x) = 0$ for all $x \in B_0$;

(iii) there is $\theta > 0$ such that if $(\lambda_n)$ is a sequence of positive numbers with $\Sigma_n \lambda_n = 1$ then there is $\alpha > 0$ and a equicontinuous sequence $(g_n)$ of members of $X^*$ for which given n and $w \in L(g_n)$ we have

$$s_B(\Sigma_n \lambda_n (g_n - w)) = \alpha, \quad s_B\left(\sum_{i=1}^{n} \lambda_i \, (g_i - w)\right) < \alpha\left(1 - \theta\left(\sum_{n+1}^{\infty} \lambda_i\right)\right);$$

(iv) there exists a continuous linear functional on X which does not attain its supremum on B.

<u>An application of James' theorem</u>: We present, as an illustration of James' characterization of reflexivity, the following result of J. Holub (see also [14]):

<u>Theorem 6</u>: Let X, Y be reflexive Banach spaces one of which possesses the approximation property. TFAE:

(i) $\mathcal{L}(X; Y)$ is reflexive;

(ii) every linear continuous operator $T: X \to Y$ is compact;

(iii) every linear continuous operator $T: X \to Y$ achieves its norm, i.e., for any linear continuous operator $T: X \to Y$ there is an $x \in X$, $\|x\| \le 1$ such that $\|Tx\| = \|T\|$.

<u>Proof</u>:

We denote by $K(X; Y)$, $I(X; Y)$, $N(X; Y)$ the compact, integral and nuclear linear operators from X to Y. By reflexivity, the possession by X( or Y) of the approximation property insures the possession of this property by the dual space $X^*$ (respectively, $Y^*$). In either case we have the identification $K(X; Y) = X^* \check{\otimes} Y$ (injective tensor product).

By duality we get

$$K(X; Y)^* = (X^* \check{\otimes} Y)^* = I(X^*; Y^*).$$

By reflexivity of $X^*$, $Y^*$ (actually we only need one of these to be reflexive at this point) and the Dunford-Pettis-Phillips Theorem

$$I(X^*; Y^*) = N(X^*; Y^*).$$

Now $X^*$ or $Y^*$ has the approximation property and is reflexive; therefore,



as any dual space possessing the so-called Radon-Nikodym property--reflexive spaces do possess this property as we shall see later--and the approximation property possesses the metric approximation property, we get that $X^*$ or $Y^*$ possesses the metric approximation property. Thus,

$$N(X^*; Y^*) = X^{**} \hat{\otimes} Y^* \text{ (projective tensor product)}$$
$$= X \hat{\otimes} Y^*.$$

Thus,

$$K(X, Y)^* = X \hat{\otimes} Y^*.$$

Finally, by the Universal Mapping Principle,

$$(X \hat{\otimes} Y^*)^* = \mathcal{L}(X; Y^{**})$$
$$= \mathcal{L}(X; Y).$$

The resulting equivalence of (i) and (ii) is now clear.

Suppose now that (ii) holds; let $T: X \to Y$ be a linear continuous operator, i.e., by (ii) T is a compact linear operator. Then there exists $(x_n) \subset X$, $\|x_n\| \le 1$ such that $\|Tx_n\| \to \|T\|$. X's reflexivity implies the existence of a subsequence of $(x_n)$--say $(x_{n_k})$--such that $x_{n_k}$ converges weakly to some $x_0$. Note $\|x_0\| \le 1$. Also T is compact hence completely continuous (i.e., T maps weakly convergent sequences into norm convergent sequences). Thus $(Tx_{n_k})$ is a norm convergent sequence in Y; in fact, note that $Tx_{n_k} \to Tx_0$ in norm. Thus

$$\|T\| = \lim_k \|Tx_{n_k}\| = \|Tx_0\|,$$

(iii) follows.

Now suppose that each linear continuous operator $T: X \to Y$ achieves its norm. Keep in mind that $\mathcal{L}(X; Y) = (X \hat{\otimes} Y^*)^*$. We will show that each linear continuous operator T, viewed as a member of $(X \hat{\otimes} Y^*)^*$, achieves its norm. Thus $(X \hat{\otimes} Y^*)$ is reflexive and, therefore, $(X \hat{\otimes} Y^*)^* = \mathcal{L}(X; Y)$ is also reflexive.