Lecture Notes in
Control and
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Edited by A.V. Balakrishnan and M. Thoma



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Berc Rustem

# Projection Methods in Constrained Optimisation and Applications to Optimal Policy Decisions

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ABSTRACT

This work is concerned with projection methods in constrained optimisation and the application of projection techniques to policy optimisation problems. The constrained optimisation problem of minimising a nonlinear function of n variables subject to equality and inequality constraints is also known as the nonlinear programming problem. Projection methods for constrained minimisation involve projections of descent directions. The basic idea underlying these methods is the principle of projections which may be considered to be the generalisation of the fact that in n-dimensional Euclidean space the shortest vector from a point to a subspace is orthogonal to the subspace.

In effect, this work consists of two parts. The first part, Chapters 1 - 4, is concerned with the development of projection techniques for different aspects of constrained optimisation. Chapter 1 provides a unified approach to the derivation of projection techniques and reviews existing methods. The application of projection techniques to the computation of a feasible point of a linearly constrained region is discussed in Chapter 2. In Chapter 3 a projection method is discussed for the quadratic programming problem of minimising a quadratic objective function subject to inequality constraints. This method requires an initial feasible point. In Chapter 4 a method is discussed for solving the nonlinear programming problem. This method requires solutions to quadratic minimisation subproblems.

The second part of the work is concerned with the application of projection techniques to computational problems in policy optimisation. A fundamental problem in the optimisation of policy decisions is the specification of a suitable objective function. In Chapter 5 an iterative method is given for specifying objective functions. In Chapter 6 policy optimisation algorithms, based on extensions of projection methods in Chapter 1 , are discussed. These algorithms minimise quadratic objective functions subject to large nonlinear models treated as equality constraints.

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## CONVENTIONS AND SYMBOLS

Assumptions, Definitions, Lemmas, Theorems etc. are given a number preceded by the chapter number. Equations and algorithms are given a number, preceded by the section and chapter numbers in which they occur.

The end of a proof or a particular train of thought is denoted by □ . The symbol $\triangleq$ denotes 'defined equal to'.

Since each chapter deals with a different aspect of constrained optimisation (e.g. feasible point algorithms, quadratic programming, nonlinear programming, etc.) the symbols used in each chapter have been slightly adapted to the needs of that chapter. The usage of commonly used symbols is given below.

Symbol

| | |
|---|---|
| $\underline{a}$ | coefficient vector of the linear term of the quadratic function |
| $\underline{b}$ | right hand side vector of a system of linear equalities and/or inequalities |
| $\underline{d}$ | descent direction |
| $E^n$ | Euclidean n-space |
| F | policy maker's admissible region for $\underline{U}$ and $\underline{Y}$ |
| $\underline{F}(\underline{Y},\underline{U}) = \underline{0}$ | Equations of the econometric model |
| $f(\underline{x})$ | nonlinear objective function , $\underline{x} \in E^n$ |

| | |
|---|---|
| $G$ | Hessian matrix |
| $G(\underline{U})$ | reduced form of $J(\underline{Y},\underline{U})$ with $\underline{Y}$ eliminated |
| $\underline{g}(\underline{U})$ | reduced form equations of the econometric model |
| $\underline{g}(\underline{x})$ | vector of inequality and/or equality constraints |
| $H$ | inverse Hessian matrix |
| $\underline{h}(\underline{x})$ | vector of equality constraints |
| $I(\underline{x})$ | index set of active constraints at $\underline{x}$ |
| $J(\underline{Y},\underline{U})$ | quadratic performance index for optimal control |
| $L(\underline{x},\underline{\lambda})$ | Lagrangian function |
| $N$ | matrix of constraint normals |
| $P, P[H], \bar{P}, \hat{P}$ | projection operators |
| $q, \bar{q}$ | quadratic functions approximating $f(\underline{x})$ |
| $\hat{q}$ | $\bar{q}$ with approximate Hessian |
| $R$ | the feasible region |
| $S(\underline{x})$ | index set of the satisfied constraints at $\underline{x}$ |
| $\underline{U}$ | vector of controls (policy instruments) |
| $V(\underline{x})$ | index set of the violated constraints at $\underline{x}$ |
| $\underline{x}_u$ | the point reached along an unconstrained direction |
| $\underline{x}_p$ | the projection of $\underline{x}_u$ |
| $\underline{Y}$ | vector of outputs (endogenous variables) |

| | |
|---|---|
| $\alpha$ | steplength factor along an unconstrained direction |
| $\underline{\gamma}$ | change in the gradient vector of the objective function |
| $\underline{\theta}(\underline{x})$ | residual vector of a system of linear equalities and/or inequalities evaluated at $\underline{x}$ |
| $\underline{\lambda}$ | the multiplier vector |
| $\tau$ | steplength along a projected direction |
| $\nabla_z(\cdot)$ | gradient vector of (.) with respect to z ( z is omitted wherever the argument is obvious) |

CONTENTS

# CHAPTER 1

# NONLINEARLY CONSTRAINED OPTIMISATION TECHNIQUES BASED ON PROJECTIONS

## 1.1 INTRODUCTION

This work is concerned with projection methods for constrained optimisation and the application of projection techniques to policy optimisation problems.

The principle of projections may be considered to be the generalisation of the fact that in n-dimensional Euclidean space the shortest vector from a point to a subspace is orthogonal to the subspace. Projections have a wide application in mathematics and especially in systems theory. For example, in the analysis of distributed systems, the method of orthogonal projections is applied by Mikhlin (1964) to solve the Dirichlet problem. Another important application arises in the derivation of the least-squares, minimum variance and recursive estimation (Kalman filtering) techniques in statistical estimation (see, e.g. Davis (1977), Luenberger (1969, Chapter 4)). Such techniques have found wide application in engineering and econometrics (see, e.g. Rustem and Velupillai (1978, 1979)). In all these applications a suitable normed linear vector space such as a Hilbert or Euclidean space is chosen and the projection of vectors in this space onto a