Stability and Control: Theory, Methods and Applications
Volume 18

# Stability and Stabilization of Nonlinear Systems with Random Structure

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# Stability and Stabilization of Nonlinear Systems with Random Structure

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# Introduction to the Series

The problems of modern society are both complex and interdisciplinary. Despite the apparent diversity of problems, tools developed in one context are often adaptable to an entirely different situation. For example, consider Lyapunov's well known second method. This interesting and fruitful technique has gained increasing significance and has given a decisive impetus for modern development of the stability theory of differential equations. A manifest advantage of this method is that it does not demand the knowledge of solutions and therefore has great power in application. It is now well recognized that the concept of Lyapunov-like functions and the theory of differential and integral inequalities can be utilized to investigate qualitative and quantitative properties of nonlinear dynamic systems. Lyapunov-like functions serve as vehicles to transform the given complicated dynamic systems into a relatively simpler system and therefore it is sufficient to study the properties of this simpler dynamic system. It is also being realized that the same versatile tools can be adapted to discuss entirely different nonlinear systems, and that other tools, such as the variation of parameters and the method of upper and lower solutions provide equally effective methods to deal with problems of a similar nature. Moreover, interesting new ideas have been introduced which would seem to hold great potential.

Control theory, on the other hand, is that branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object implies the influence of its behavior so as to accomplish a desired goal. In order to implement this influence, practitioners build devices that incorporate various mathematical techniques. The study of these devices and their interaction with the object being controlled is the subject of control theory. There have been, roughly speaking, two main lines of work in control theory which are complementary. One is based on the idea that a good model of the object to be controlled is available and that we wish to optimize its behavior, and the other is based on the constraints imposed by uncertainty about the model in which the object operates. The control tool in the latter is the use of feedback in order to correct for deviations from the desired behavior. Mathematically, stability theory, dynamic systems and functional analysis have had a strong influence on this approach.

Volume 1, *Theory of Integro-Differential Equations*, is a joint contribution by V. Lakshmikantham (USA) and M. Rama Mohana Rao (India).

Volume 2, Stability Analysis: Nonlinear Mechanics Equations, is by A.A. Martynyuk (Ukraine).

Volume 3, *Stability of Motion of Nonautonomous Systems: The Method of Limiting Equations*, is a collaborative work by J. Kato (Japan), A.A. Martynyuk (Ukraine) and A.A. Shestakov (Russia).

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Volume 18 *Stability and Stabilization of Nonlinear Systems with Random Structure* by I.Ya. Kats (Russia) & A.A. Martynyuk (Ukraine).

Due to the increased interdependency and cooperation among the mathematical sciences across the traditional boundaries, and the accomplishments thus far achieved in the areas of stability and control, there is every reason to believe that many breakthroughs await us, offering existing prospects for these versatile techniques to advance further. It is in this spirit that we see the importance of the "Stability and Control" series, and we are immensely thankful to Taylor & Francis for their interest and cooperation in publishing this series.

# Preface

Probability models are widely used in the investigation of real processes occurring in nature and technology. This involves consideration of ordinary differential equations, the parameters of which are random time functions. Since stability is one of the main conditions of the process physical realization, analysis of probability models has resulted in the corresponding development of a general theory of motion stability known as stochastic stability theory.

The Lyapunov function (functionals) method has proved to be a most versatile and powerful approach in the investigation of the stability of determined systems. The main advantage of this method is that the stability of the system can be assessed without immediate integration of the differential motion equations.

The insuperable difficulties associated with the integration of stochastic systems mean that the stability theory to such systems is based on the fundamental ideas of the Lyapunov functions method.

To construct an efficient stability theory of stochastic systems the following are of prime importance:

- consideration of systems possessing Markov properties (absence of after-effects);
- concept of strong probability stability of the asymptotic behaviour of the process realizations;
- use of the auxiliary Lyapunov functions and, in particular, the notion of the auxiliary function derivative by virtue of the system whose computation does not require integration of the stochastic perturbed motion equations.

The abovementioned approach for systems with random Markov parameters was proposed by Krasovskii in 1960 and set forth by Kats and Krasovskii. The ensuing intensive development of this approach related mainly to investigation of the stability of solutions of stochastic differential Ito equations. The results have been presented in many well-known and important monographs and textbooks. However, many experts in this area were unaware of investigations on the stability of systems with random Markov parameters because the papers were published in different journals. This has also been the case for problems on the stabilization of controlled motions for which the results for systems with random parameters were also published in the early 1960s in journal papers by Krasovskii and Lidskii.

Our aim is to fill this gap and we hope that this monograph will be of valuable in this field help to experts and beginners alike.

This book consists of six chapters. The first chapter covers background material and the ideas of the theory of probability processes. Some examples from mechanics show the essence of the process of modelling real phenomena using systems of ordinary differential equations with random structure.

Chapter 2 sets out the method of analysis of the stochastic stability of systems with random structure. The method is based on scalar Lyapunov functions and the idea of an averaged derivative of an auxiliary function along solutions of the stochastic system.

In Chapter 3 describes the analysis of stochastic stability based on Lyapunov's matrix function method. Here stochastic singularly perturbed systems are studied together with systems of ordinary differential equations with random parameters.

Chapter 4 presents the results of the stability analysis of systems with random structure using the Lyapunov function constructed for the first approximation of a nonlinear system.

Chapter 5 discusses the problem of constructing systems with a prescribed type of motion stability for the optimal of the transient process.

The concluding chapter shows how efficient stability criteria for solutions of stochastic models of real systems can be obtained using the method of matrix-valued Lyapunov functions.

Thus, this monograph develops the theory of stochastic stability and stabilization for objects modelled using differential equations the parameters of which are Markov functions of a quite general nature.

The assumption that at random times of jump-like change of system parameters the phase vector of its state can also change in a jump-like way is new. Such systems are called systems with random structure. This term is different to that used in some publications on the theory of automatic control and describes a wider class of systems.

Our approach to discontinuous phase trajectories in this monograph differs from the model of generalized stochastic differential Ito or Poisson equations, and is in our opinion an efficient one.

The idea of writing this monograph came from Professor A.A.Martynyuk. The book describes work done during 1960 – 1999.

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I. Ya. Kats

A. A. Martynyuk

# Contents

# 1 Preliminary Analysis

## 1.0 Introductory Remarks

This chapter forms an introduction. It presents the main ideas of the theory of probability processes, particularly Markov processes, which are used in this book. The various types of stochastically differential equations are discussed and the possibilities for their use in modelling systems subject to random perturbations. In particular, perturbations are considered that result in random changes of the system structure.

Sections 1.1 and 1.2 set out standard information from the probability theory and theory of random processes including the methods of description of pure discontinuous Markov processes and Markov chains with a finite number of states.

Section 1.3 describes the known standard probability processes such as the Wiener and Poisson processes. On this basis stochastic differential Ito and Poisson equations are considered. Examples of mathematical models described by these types of equation are cited.

Section 1.4 deals with systems of ordinary differential equations the parameters of which are random Markov processes. The various behaviours of the phase trajectories at the times of random jump-like change of the system parameters are discussed. A class of systems with random structure is determined and the constructive methods of description of such systems are proposed. Concrete problems of mechanics and other problems are considered which involve the necessity to study systems with random structure.

## 1.1 Random Variables and Probability Distributions

This section deals with the main terms and fundamental ideas of probability theory. They will be used in subsequent sections without additional references.

The reader is assumed to have be familiar with the following:

(1) the random variable $\xi$;

(2) the *distribution function* $F_\xi(x)$ of the random variable $\xi$ determined as $F_\xi(x) = P\{\xi < x\}$ and its fundamental properties, ($P(A)$ is the probability of event $A$);

(3) events related to the values of the random variable $\xi$ and their probabilities;

(4) mathematical expectation $E\xi$ of the random variable and higher order moments $E\xi^k$;

(5) the complete probability formula and Byes formula.

The *random variable* $\xi$ is called *discrete* if there exists a finite or countable set of different numbers $x_1, x_2, \ldots,$ such that $p_i = P\{\xi = x_i\} > 0$, $i = 1, 2, \ldots,$ and $\sum p_i = 1$.

If there exists a nonnegative function $p_\xi(x)$ defined on the whole axis $-\infty < x < +\infty$ such that the distribution function $F_\xi(x)$ can be represented as

$$F_\xi(x) = \int_{-\infty}^{x} p_\xi(t)\,dt, \tag{1.1.1}$$

then $p_\xi(x)$ is called a *distribution density*, and the random variable $\xi$ is continuous.

If $\xi$ is a discrete random variable, then

$$E\xi^k = \sum_i x_i^k p_i, \tag{1.1.2}$$

providing the series (1.1.2) absolutely converges.

The moments for continuous random variables are determined by the correlation

$$E\xi^k = \int_{-\infty}^{+\infty} x^k p_\xi(x)\,dt, \tag{1.1.3}$$

provided that the integral (1.1.3) is absolutely convergent.

The *dispersion* $D\xi = \sigma_\xi^2$ *of the random variable* is defined as

$$D\xi = \sigma_\xi^2 = E(\xi - E\xi)^2 \tag{1.1.4}$$

and $\sigma_\xi = \sqrt{D\xi}$ is called the *mean square deviation*.