REPRESENTATION THEORY OF THE SYMMETRIC GROUP

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PREFACE

THE ordinary representation theory of a finite group was largely developed by Frobenius, Burnside and Schur, and the modular theory by Dickson, Brauer and Nesbitt. In the first chapter of this book we shall try to provide the background for those parts of the subject which are necessary in the sequel. With regard to the ordinary theory, a fairly complete though brief account of these 'classical' ideas is given in Part 1. The case of the modular theory is somewhat different, since new and fundamental developments are still in progress. Our purpose in Part 2 is to present as clear a picture as possible of a complicated piece of mathematics. Though proofs are omitted in §§ 12.5 and 12.6, corresponding theorems will be proved later on for $\mathfrak{S}_n$ by quite different methods and in a more explicit form. Alternative approaches to these general ideas are available, and we have chosen that due to Osima and Nagao in which the ordinary and modular theories appear as different aspects of an integrated whole.

The ordinary representation theory of the symmetric group $\mathfrak{S}_n$ was first developed by Frobenius. Just a year later (1900-1) an independent approach was given by Alfred Young which was based on a study of the group algebra and its idempotents. Young was primarily interested in applications to projective invariants and his work on representation theory is scattered through a long series of papers. D. E. Rutherford has collected together this material and the reader is referred to his *Substitutional Analysis* for an account of Young's work. Young's Fundamental Theorem (2.17) giving the actual matrices which generate any given irreducible representation is quoted without proof, as is also his substitutional equation (2.23) which leads to the definition of the *raising operator*. From this point on the presentation is self-contained.

The interesting aspect of the theory is the crucial role played at every stage by the Young diagram $[\lambda]$. One may legitimately ask why so many concepts, often very involved when stated in general terms, become so simple when applied to $\mathfrak{S}_n$ and interpreted with reference to $[\lambda]$. That this is so provides the chief reason for the writing of the present book, since it is conceivable that a corresponding approach through the group algebra may lead to a simplification and clarification of the general theory. Some slight progress along these lines has already been made. Fundamentally, the problem is to derive the representation theory of a given group in terms of that of suitably chosen subgroups.

In the case of $\mathfrak{S}_n$ such suitable sub-groups are easily recognized to be of the form

$$\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \ldots \times \mathfrak{S}_{\lambda_h}, \quad \sum \lambda_i = n.$$

Many authors have contributed to the theory described here as a glance at the Bibliography will show. Particular mention should be made of the work of J. S. Frame, D. E. Littlewood, Masaru Osima and R. M. Thrall. Notes on the various sections with appropriate references will be found at the end of the book, but the method of presentation is often changed in an attempt to coordinate the work of different authors. The greater part of the material in Chapters VII and VIII is published here for the first time and is based on theses by O. E. Taulbee and Diane Johnson. The ideas are complicated and it may be that further work will lead to significant simplifications, but the general pattern of development is clear.

The chief feature of this account of the representations of $\mathfrak{S}_n$ is the use made of Young's *raising operator* which plays a major role in Chapters II and III and again in Chapters VI, VII and VIII. That it is possible to express the reduction of the appropriate permutation representations of $\mathfrak{S}_n$ in an explicit manner seems to be largely responsible for the completeness of the theory.

In conclusion, I would like to express my thanks to Professor J. S. Frame, Professor Masaru Osima, and to my colleague Professor A. J. Coleman for reading the manuscript and making many useful suggestions. I am particularly indebted to Professor Hirosi Nagao for his friendly interest and valuable criticism of Chapters I, VII and VIII. But above all I am indebted to my wife for her continued encouragement over many years.

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CHAPTER ONE

THE ORDINARY AND THE MODULAR REPRESENTATION THEORY OF A FINITE GROUP

PART 1: THE ORDINARY REPRESENTATION THEORY

Introduction. We propose to set out here those parts of the representation theory of a finite group over the complex field which are essential in what follows. While such a survey is necessarily incomplete, two aspects of the theory can be emphasized:

(*a*) It is important to make clear just how far character theory goes: that it is a *class* theory, providing an explicit criterion of irreducibility but no information concerning the irreducible representations beyond equivalence.

(*b*) The *subgroup structure* of a finite group $\mathcal{G}$ is largely untouched as yet by character theory. Yet certain connections can be made and Frobenius' reciprocity theorem is vital in this regard. The fact that we have so little knowledge of the irreducible components of a permutation representation of $\mathcal{G}$ stands out.

The significance of these deficiencies of the general theory is greatly clarified by a study of the representation theory of the symmetric group $\mathfrak{S}_n$. First developed by Alfred Young, it has been extended by many authors and these developments are brought together in this book with the view of describing a pattern to which any generalization of Young's theory must conform.

11.1 Permutation representations. Historically, the notion of a group arose early in the 19th century as a group of *permutations* on the roots of an algebraic equation. In this context Galois showed its usefulness in providing a criterion for the solvability of an equation by radicals. It was not till 1854 that Cayley defined an *abstract* group as a set $\mathcal{G}$ of elements G_i subject to a law of combination which we may take to be multiplication and such that:

(*a*) for every G_i, G_j in $\mathcal{G}$ there exists an element G_k in $\mathcal{G}$ such that

$$G_iG_j = G_k;$$

(*b*) $G_i(G_jG_k) = (G_iG_j)G_k$ (the associative law);

(*c*) there exists an *identity* G_1 $(=I)$ in $\mathcal{G}$ such that $G_iI = G_i = IG_i$;

(*d*) for each G_i there exists an *inverse* G_i^{-1} in $\mathcal{G}$ such that

$$G_iG_i^{-1} = I = G_i^{-1}G_i.$$

These simple properties find their realization in many different domains. For example: (i) the ordinary integers of arithmetic form a group with respect to addition, with zero as the identity element; (ii) the rotations and translations in a plane form a group as do the symmetries of a sphere (which leave the centre fixed). In what follows we shall always assume that the number of elements in $\mathscr{G}$ is *finite* and we shall call this number the *order* g of $\mathscr{G}$. In this section we propose to reverse the historical procedure and see how, with each element G_i of $\mathscr{G}$, we may associate a permutation P_i such that $P_iP_j=P_k$ as in (*a*). Such a set of permutations is said to form a *permutation representation* of $\mathscr{G}$.

To this end consider a subgroup $\mathscr{H}$ of $\mathscr{G}$ of order h. If we gather together those elements of $\mathscr{G}$ which may be written in the form HG_i, where H is any element of $\mathscr{H}$ and G_i is fixed, we may denote them $\mathscr{H}G_i$, called a right *coset* of $\mathscr{H}$. Any two such cosets of $\mathscr{H}$ are entirely distinct or coincide, so that we may write

11.11 $$\mathscr{G}=\mathscr{H}+\mathscr{H}G_2+\mathscr{H}G_3+\ldots+\mathscr{H}G_n,$$

with $n=g/h$. It follows readily that the corresponding left cosets $G_i^{-1}\mathscr{H}$ are also distinct so that

11.12 $$\mathscr{G}=\mathscr{H}+G_2^{-1}\mathscr{H}+G_3^{-1}\mathscr{H}+\ldots+G_n^{-1}\mathscr{H}.$$

Now let us multiply every coset $\mathscr{H}G_i$ in 11.11 on the right by some element of $\mathscr{G}$. The effect will be to leave $\mathscr{H}G_i$ unchanged or to change it into some other coset, thus yielding a *permutation* of the n cosets which we may denote by

$$G\rightarrow\begin{pmatrix}\mathscr{H}, & \mathscr{H}G_2, & \ldots\mathscr{H}G_n\\ \mathscr{H}G, & \mathscr{H}G_2G, & \ldots\mathscr{H}G_nG\end{pmatrix}=\begin{pmatrix}\mathscr{H}G_i\\ \mathscr{H}G_iG\end{pmatrix}.$$

Since

$$\begin{pmatrix}\mathscr{H}G_i\\ \mathscr{H}G_iG\end{pmatrix}\begin{pmatrix}\mathscr{H}G_iG\\ \mathscr{H}G_iGG'\end{pmatrix}=\begin{pmatrix}\mathscr{H}G_i\\ \mathscr{H}G_iGG'\end{pmatrix},$$

these permutations form a *representation* $\mathscr{G}^{\mathscr{H}}$ of $\mathscr{G}$.

Of course it may happen that $\mathscr{H}G_iG=\mathscr{H}G_i$ for all i, in which case $G_iGG_i^{-1}\in\mathscr{H}$ and $\mathscr{H}$ contains a largest subgroup $\mathscr{K}$ which is normal in $\mathscr{G}$. The mapping $\mathscr{G}\rightarrow\mathscr{G}^{\mathscr{H}}$ is a *homomorphism* with $\mathscr{G}^{\mathscr{H}}$ *isomorphic to the factor group* $\mathscr{G}\,/\,\mathscr{K}$.

If we multiply the cosets in 11.12 on the left by G^{-1} we similarly obtain a permutation

$$G\rightarrow\begin{pmatrix}\mathscr{H}, & G_2^{-1}\mathscr{H}, & \ldots & G_n^{-1}\mathscr{H}\\ G^{-1}\mathscr{H}, & G^{-1}G_2^{-1}\mathscr{H}, & \ldots & G^{-1}G_n^{-1}\mathscr{H}\end{pmatrix}=\begin{pmatrix}G_i^{-1}\mathscr{H}\\ G^{-1}G_i^{-1}\mathscr{H}\end{pmatrix}.$$

and so again a representation of $\mathcal{G}$. In view of the isomorphism

11.13 $$\begin{pmatrix} G_i^{-1}\mathcal{H} \\ \mathcal{H}G_i \end{pmatrix} \begin{pmatrix} \mathcal{H}G_i \\ \mathcal{H}G_iG \end{pmatrix} \begin{pmatrix} \mathcal{H}G_i \\ G_i^{-1}\mathcal{H} \end{pmatrix} = \begin{pmatrix} G_i^{-1}\mathcal{H} \\ G^{-1}G_i^{-1}\mathcal{H} \end{pmatrix},$$

these right and left permutation representations are *formally equivalent*. Just as we replaced each group element by its inverse, and the coset $\mathcal{H}G_i$ by $G_i^{-1}\mathcal{H}$ in 11.13, so we may replace $\mathcal{H}$ by any conjugate subgroup $G^{-1}\mathcal{H}G$ and arrive at a formally equivalent permutation representation. Thus we may speak of the permutation representation $\mathcal{G}^{\mathcal{H}}$ of $\mathcal{G}$ *induced* by the subgroup $\mathcal{H}$.

If $\mathcal{H}=I$ we have a particularly important permutation representation of $\mathcal{G}$ of degree g called the *regular* representation, since no symbol remains fixed except under the identity and every cycle in any permutation is of equal length. The left regular representation has the remarkable property of containing all permutations which commute with every permutation of the right regular representation and *vice versa*, since

$$\begin{pmatrix} G \\ GG_i \end{pmatrix}\begin{pmatrix} G \\ G_j^{-1}G \end{pmatrix} = \begin{pmatrix} G \\ G_j^{-1}GG_i \end{pmatrix} = \begin{pmatrix} G \\ G_j^{-1}G \end{pmatrix}\begin{pmatrix} G \\ GG_i \end{pmatrix}.$$

There is a second way of constructing a permutation representation of $\mathcal{G}$ which is important in the sequel. If $G^{-1}\mathcal{K}G=\mathcal{K}$ then the totality of such elements constitutes the *normalizer* $\mathcal{N}(\mathcal{K})$ of $\mathcal{K}$ and it can easily be seen that the number of distinct conjugate subgroups of $\mathcal{K}$ is equal to the index of $\mathcal{N}(\mathcal{K})$ in $\mathcal{G}$. If we now *transform* these conjugate subgroups by G they will be permuted amongst themselves according to the permutation

$$G \to \begin{pmatrix} G_i^{-1}\mathcal{K}G_i \\ G^{-1}(G_i^{-1}\mathcal{K}G_i)G \end{pmatrix}.$$

Again, such permutations form a representation of $\mathcal{G}$ which is equivalent to $\mathcal{G}^{\mathcal{H}}$ if we set $\mathcal{H}=\mathcal{N}(\mathcal{K})$. If $\mathcal{K}$ is normal in $\mathcal{G}$ (e.g. if $\mathcal{K}=I$) this representation by transformation collapses. However, in the case of the symmetric group $\mathfrak{S}$ we shall always be considering subgroups of the form

$$\mathcal{K} = \mathfrak{S}_p \times \mathfrak{S}_q \times \ldots \times \mathfrak{S}_r \qquad p+q+\ldots+r = n,$$

where $\times$ indicates the 'direct product' and each factor affects different symbols. By keeping track of the order of the factors and the symbols involved we can obtain a permutation representation equivalent to $\mathcal{G}^{\mathcal{K}}$. These ideas will be developed in detail in the following chapter.

Of course, one can conclude from these constructions that *every* finite group is isomorphic to a subgroup of $\mathfrak{S}_n$ for some value of n, but this does not seem to be a very fruitful line of thought.

From any permutation representation $\mathscr{G}^{\mathscr{H}}$ we obtain immediately a *linear* representation of $\mathscr{G}$ by permutation matrices (G) having one 1 in each row and column and zeros elsewhere. The rule for constructing (G) depends on our convention for multiplying permutations. Since permutations are operators we shall multiply them from *right to left* and place 1 at the intersection of the ith row and jth column of (G), when $\mathscr{H}G_iG=\mathscr{H}G_j$. One may, however, multiply permutations from left to right; with such a convention the matrix (G) would be replaced by its transpose $(G)'$, where

$$(G)' = (G)^{-1} = (G^{-1}) .$$

The number of 1's in the diagonal of (G) is called the *permutation character* of G and written $\chi^{\mathscr{H}}(G)$. If g_G is the number of conjugates of G in $\mathscr{G}$ and $g_G^{\mathscr{H}}$ the number of these that lie in $\mathscr{H}$:

11.14
$$\chi^{\mathscr{H}}(G) = \frac{gg_G^{\mathscr{H}}}{hg_G}.$$

Proof. For each 1 in $\chi^{\mathscr{H}}(G)$ we must have $\mathscr{H}G_iG=\mathscr{H}G_i$ so that $G_iGG_i^{-1}\in\mathscr{H}$. If G_i runs over *all* elements of $\mathscr{G}$ we obtain a given conjugate in $\mathscr{H}$ counted g/g_G times; but there are $g_G^{\mathscr{H}}$ such conjugates in $\mathscr{H}$ so we must multiply g/g_G by $g_G^{\mathscr{H}}$. On the other hand if $G_iGG_i^{-1}\in\mathscr{H}$ so also $(HG_i)G(HG_i)^{-1}\in\mathscr{H}$ for every $H\in\mathscr{H}$; if each such solution to the problem is to be counted once only we must divide $gg_G^{\mathscr{H}}/g_G$ by h, proving the theorem.

We state the following two relations between permutation characters:

11.15
$$\sum_G \chi^{\mathscr{H}}(G) = g,$$

11.16
$$\sum_G \chi^{\mathscr{H}}(G)\,\chi^{\mathscr{K}}(G) = tg,$$

where t is the number of double cosets in the decomposition

$$\mathscr{G} = \mathscr{H}\mathscr{K} + \mathscr{H}G_2\mathscr{K} + \ldots + \mathscr{H}G_t\mathscr{K}.$$

It is an interesting and easy exercise to construct the proof of 11.15 but that of 11.16 is more difficult.

11.17 *Example.* By way of illustration we take $\mathscr{G}=\mathfrak{S}_4$ and list below a representative set of subgroups.

TABLE I

$\mathscr{H}$	*Subgroup*	h
$\mathscr{H}_1$	I	1
$\mathscr{H}_2$	I, (12)	2
$\mathscr{H}_3$	I, (12) (34)	2
$\mathscr{H}_4$	I, (123), (132)	3
$\mathscr{H}_5$	I, (1324), (12) (34), (1423)	4
$\mathscr{H}_6$	I, (12) (34), (14) (23), (13) (24)	4
$\mathscr{H}_7$	I, (12), (34), (12) (34)	4
$\mathscr{H}_8$	I, (12), (13), (23), (123), (132)	6
$\mathscr{H}_9$	I, (12), (34), (12) (34), (14) (23), (13) (24), (1324), (1423)	8
$\mathscr{H}_{10}$	$\mathfrak{A}_4$	12
$\mathscr{H}_{11}$	$\mathfrak{S}_4$	24

In particular, we may write the cosets of the subgroup $\mathscr{H}_9$ in the form

11.171 $$\mathfrak{S}_4 = \mathscr{H}_9 + \mathscr{H}_9\,(23) + \mathscr{H}_9\,(24).$$

Since $\mathscr{H}_9 \supset \mathscr{H}_6$ which is normal in $\mathfrak{S}_4$, the permutation representation $\mathscr{G}_9$ (cf. Table II) is isomorphic to $\mathfrak{S}_4 / \mathscr{H}_6 \sim \mathfrak{S}_3$. Denoting the three cosets of $\mathscr{H}_9$ by b, c, d respectively, we have

$$(23) \rightarrow (bc) \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (34) \rightarrow (cd) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which generate the representation.

The permutation characters are listed in Table II below, in which $\mathscr{G}_i$ is used as an abbreviation for $\mathscr{G}^{\mathscr{H}_i}$.

TABLE II—$\chi_i^{\mathscr{H}_j}$

Class	(1^4)	$(2, 1^2)$	(2^2)	$(3, 1)$	(4)
g_i	1	6	3	8	6
$\mathscr{G}_1$	24	.	.	.	.
$\mathscr{G}_2$	12	2	.	.	.
$\mathscr{G}_3$	12	.	4	.	.
$\mathscr{G}_4$	8	.	.	2	.
$\mathscr{G}_5$	6	.	2	.	2
$\mathscr{G}_6$	6	.	6	.	.
$\mathscr{G}_7$	6	2	2	.	.
$\mathscr{G}_8$	4	2	.	1	.
$\mathscr{G}_9$	3	1	3	.	1
$\mathscr{G}_{10}$	2	.	2	2	.
$\mathscr{G}_{11}$	1	1	1	1	1

The headings indicate the cycle structure of each *class* of permutations followed by the number g_i of elements in the class. Using 11.14 it is easy to check the construction of the table.

11.2 The group algebra over a field F. If we take the elements $G_1(=I)$, $G_2, \ldots, G_g$ of the group $\mathscr{G}$ as basis elements of a *group algebra* $\mathscr{A}$ with coefficients in a given field F, we can define addition and multiplication in the algebra as follows:

11.21
$$\begin{aligned}(\sum x_iG_i)+(\sum y_iG_i) &= \sum (x_i+y_i)G_i,\\ (\sum x_iG_i)(\sum y_jG_j) &= \sum (x_iy_j)G_iG_j.\end{aligned}$$

Such a group algebra is clearly associative with a unity element. If we denote any element of $\mathscr{A}$ by

11.22
$$a = \sum_{i=1}^{g} x_iG_i,$$

then

11.23
$$a \,.\, G_i = \sum_{j=1}^{g} s_{ji}G_j, \qquad G_i \,.\, a = \sum_{j=1}^{g} r_{ij}G_j,$$

and the matrices (s_{ji}) and $(r_{ij})'$ constitute the *left* and *right regular* representations of $\mathscr{A}$ where $(r_{ij})'$ denotes the transpose of (r_{ij}). A different choice of basis elements leads to an *equivalent* regular representation. In general, if the two regular representations are equivalent the algebra is called a *Frobenius algebra*; this condition is certainly satisfied in the case of a group algebra, since

$$G_i^{-1} \,.\, a = \sum_{j=1}^{g} s_{ij}G_j^{-1}$$

for a choice of new basis elements G_i^{-1} $(i=1, 2, \ldots, g)$.

The *centre* of an algebra $\mathscr{A}$ is made up of all those elements which commute with every element of $\mathscr{A}$. To describe such elements, let us denote the classes of conjugate elements of $\mathscr{G}$ by

$$C_1\ (=I),\ C_2, \ldots, C_k,$$

where there are g_i elements in C_i and $g=g_1+g_2+\ldots+g_k$. No confusion will arise if we use the same symbol C_i to denote the *sum* of conjugate elements in $\mathscr{A}$.

11.24 *The necessary and sufficient condition that an element a belongs to the centre of $\mathscr{A}$ is that it have the form*

$$a = \sum_{i=1}^{k} y_iC_i.$$

Proof. If $ab=ba$ for all b then $G_i a = aG_i$ for all i, so all conjugates of a given group element must appear in a with the same coefficient. Clearly, the condition is sufficient.

Multiplication of the sets C_i is also important, and a similar argument leads to the relation

11.25
$$C_l C_m = C_m C_l = \sum_{n=1}^{k} c_{lmn} C_n.$$

where the $c_{lmn} = c_{mln}$ are integers.

If we set all $x_i = 0$ in 11.22 we obtain the *zero element* 0 of $\mathscr{A}$. If $a^n = 0$ for some index n then a is *nilpotent*. On the other hand, an element a' of $\mathscr{A}$ is said to be *properly nilpotent* if aa', and so $a'a$, is nilpotent for every a in $\mathscr{A}$. Thus the properly nilpotent elements of $\mathscr{A}$ form an *ideal* of $\mathscr{A}$ called the *radical*. It is customary to denote the radical by $\mathscr{N}$ so that the quotient algebra $\mathscr{A} / \mathscr{N}$ is *semi-simple*, i.e. can be written as the direct sum of *simple* algebras $\mathscr{A}_i$

$$\mathscr{A} / \mathscr{N} \sim \mathscr{A}_1 + \mathscr{A}_2 + \ldots$$

11.26 *Example.* Consider the algebra $\mathscr{A}$ of all 3×3 matrices (a_{ij}) with $a_{12} = a_{13} = a_{23} = 0$, the remaining coefficients being arbitrary elements of the field F. Then

$$\mathscr{N}: \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}, \qquad \mathscr{N}^2: \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix},$$

where $\mathscr{N}^2$ is defined to be the set of all products of any two elements of $\mathscr{N}$. $\mathscr{A} / \mathscr{N} \sim \mathscr{A}_1 + \mathscr{A}_2 + \mathscr{A}_3$, where

$$\mathscr{A}_1: \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathscr{A}_2: \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathscr{A}_3: \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

In this case $\mathscr{N}^3 = 0$, i.e. the product of any element of $\mathscr{N}$ by any element of $\mathscr{N}^2$ must be the zero matrix.

The following Theorem is of fundamental importance:

11.27 *If the field F is of characteristic zero, the group algebra $\mathscr{A}$ is semi-simple.*

Proof. From the *anti-isomorphism* of $\mathscr{A}$:

$$a = \sum_i x_i G_i \qquad a' = \sum x_i^{-1} G_i^{-1},$$

where $x_i^{-1} = 0$ if $x_i = 0$, we conclude that if $a \in \mathscr{N}$ then so also is a', and conversely. From the definition it follows that for any $a \neq 0$ in $\mathscr{N}$

there exists an integer m such that $a \in \mathcal{N}^m$ and $a' \in \mathcal{N}^m$, with $\mathcal{N}^{2m}=0$. But the coefficient of I in aa' can never vanish so that $aa' \neq 0$, which is a contradiction. Thus no $a \neq 0$ exists in $\mathcal{N}$ and $\mathcal{A}$ is semi-simple.

If F is of characteristic different from zero this proof fails and the group algebra may possess a radical, as we shall see.

There is one other important idea which we must introduce here: if $e_1^2=e_1$ then e_1 is said to be *idempotent*. If e_2 is also an idempotent such that $e_1e_2=e_2e_1=0$, then

$$(e_1+e_2)^2 = e_1+e_2$$

so that e_1+e_2 is an idempotent. Continuing thus, we may construct in at most g steps a *principal idempotent* e for which no further e' exists. That this principal idempotent is unique is easily seen. For,

$$eI = e = Ie, \quad (I-e)\,e = e\,(I-e) = 0,$$

and $e+(I-e)=I$ is also idempotent. But, by assumption, this can only happen if $I-e=0$ so that $e=I$. Thus:

11.28 *The only principal idempotent in a group algebra is the identity.*

11.3 Character theory. Consider a group $\mathcal{G}$ of finite order and a group (G) of nonsingular matrices or linear transformations of a vector-space V with coefficients in a field F, which we shall assume is algebraically closed. If (G) is homomorphic to $\mathcal{G}$, every element of a normal subgroup $\mathcal{H}$ is mapped on the identity matrix of (G); if $(G_i)(G_j)=(G_iG_j)$ for all i, j, (G) is called a *representation* of $\mathcal{G}$, and $\mathcal{G}\,/\,\mathcal{H}$ is isomorphic to (G). From such a representation of $\mathcal{G}$ we obtain a representation of the group algebra $\mathcal{A}$ by replacing G_i by its representative matrix (G_i) in 11.22. If we interpret the x_i in 11.22 as independent variables the matrix

11.31 $$G = \sum_{i=1}^{g} x_i(G_i)$$

is called the *group matrix* of the representation (G).

A representation (G) is said to be *decomposable* if the vector space V is the direct sum of two subspaces V_1 and V_2 *both* of which are invariant under (G). Adapting the coordinate system in V to this decomposition we have

11.32 $$(G) = \begin{bmatrix} (G)_1 & 0 \\ 0 & (G)_2 \end{bmatrix};$$

where $(G)_1$ operates in V_1 and $(G)_2$ in V_2. If such reduction be continued we finally obtain a splitting of (G) into *indecomposable components* $(G)_i$ which are uniquely determined up to a linear transformation. Each $(G)_i$ is also a representation of $\mathcal{G}$.

While the representation (G) may be indecomposable it may still be *reducible*, in which case the vector-space V contains a subspace V_1 which is invariant under (G), although the complementary subspace V_2 is not. Again, adapting the coordinate system in V, we have

11.33 $$(G) = \begin{bmatrix} (G)_{11} & 0 \\ (G)_{21} & (G)_{22} \end{bmatrix}$$

where $(G)_{11}$ and $(G)_{22}$ are representations of $\mathcal{G}$. If $(G)_{11}$ and $(G)_{22}$ are irreducible then clearly the *radical* of the group algebra is obtained by setting $(G)_{11}=(G)_{22}=0$ in 11.33. If the field F has characteristic zero, in particular if F is the complex field, $(G)_{21}=0$ and (G) is *completely reducible* by 11.27; in this case the concepts of indecomposability and irreducibility coincide.

The following theorem is fundamental in the theory:

11.34 SCHUR'S LEMMA. *If X and X' are two irreducible group matrices of $\mathcal{G}$ degrees f and f' and P is a constant $f\times f'$ matrix such that $XP=PX'$, then (i) $P=0$ or (ii) $f=f'$ and $|P|\neq 0$.*

Proof. Let us assume that P is a non-zero matrix of rank $r>0$, and set

$$f-r = s, \quad f'-r = t,$$

so that there exist non-singular matrices A, B such that $APB=Q$ where

$$Q = \begin{bmatrix} I & 0_{rt} \\ 0_{sr} & 0_{st} \end{bmatrix}.$$

In this matrix Q, I_{rr} is the identity matrix of degree r while 0_{rt} is the zero matrix having r rows and t columns. If we write $AXA^{-1}=X_1$, $B^{-1}X'B=X_1'$, then the relation $XP=PX'$ becomes

$$X_1Q = QX_1',$$

so that, by suitably partitioning X_1 and X_1' we may write

$$X_1Q = \begin{bmatrix} X_{rr} & X_{rs} \\ X_{sr} & X_{ss} \end{bmatrix}\begin{bmatrix} I_{rr} & 0_{rt} \\ 0_{sr} & 0_{st} \end{bmatrix} = \begin{bmatrix} I_{rr} & 0_{rt} \\ 0_{sr} & 0_{st} \end{bmatrix}\begin{bmatrix} X'_{rr} & X'_{rt} \\ X'_{tr} & X'_{tt} \end{bmatrix} = QX_1',$$

or

$$\begin{bmatrix} X_{rr} & 0_{rt} \\ X_{sr} & 0_{st} \end{bmatrix} = \begin{bmatrix} X'_{rr} & X'_{rt} \\ 0_{sr} & 0_{st} \end{bmatrix},$$

from which we conclude that $X_{sr}=0_{sr}$, $X'_{rt}=0_{rt}$. But if $r<f$ or $r<f'$ this would imply that X or X' is reducible, contrary to supposition. Thus if $P\neq 0$, $r=f=f'$ and $|P|\neq 0$, so that

$$P^{-1}XP = X'$$

and X and X' are said to be *equivalent*.

This definition of the equivalence of the group matrices X, X' includes in particular the change of basis elements introduced in § 11.2.

There is a remarkable corrollary of Schur's Lemma namely:

11.341 *If X is irreducible and $P^{-1}XP=X$ then $P=aI_f$.*

Proof. If a is a characteristic root of P then $|P-aI_f|=0$. But clearly

$$X(P-aI_f) = (P-aI_f)X$$

so that the matrix $(P-aI_f)$ must be the zero matrix by 11.34, and $P=aI_f$ as required.

This conclusion enables us to dispose of the representation theory of Abelian groups once and for all:

11.342 *Every irreducible representation of an Abelian group is of degree* 1.

Proof. If $X=\sum x_i(G_i)$ for any representation (G) of the Abelian group $\mathcal{G}$, it follows immediately that

$$X(G_i) = (G_i)X,$$

so that $(G_i)=\rho I_f$ for every i. But this is just the condition above, that every irreducible representation must be of degree 1.

There is one further application of Schur's Lemma which gives explicit information concerning the matrices of an irreducible representation.

11.343 *If $X=(x_{ij})$ and $X'=(x'_{mn})$ are the group matrices of two irreducible representations $G\rightarrow(a^G_{ij})$ of degree f and $G\rightarrow(b^G_{mn})$ of degree f' of $\mathcal{G}$, where:*

$$x_{ij} = \sum_G a^G_{ij}x_G, \qquad x'_{mn} = \sum_G b^G_{mn}x'_G,$$

then

(i) $$\sum_G a^{G^{-1}}_{ij} a^G_{kl} = \frac{g}{f}\delta_{il}\delta_{jk} \qquad (i, j, k, l = 1, \ldots, f),$$

(ii) $$\sum_G a^{G^{-1}}_{ij} b^G_{mn} = 0 \qquad (m, n = 1, \ldots, f').$$