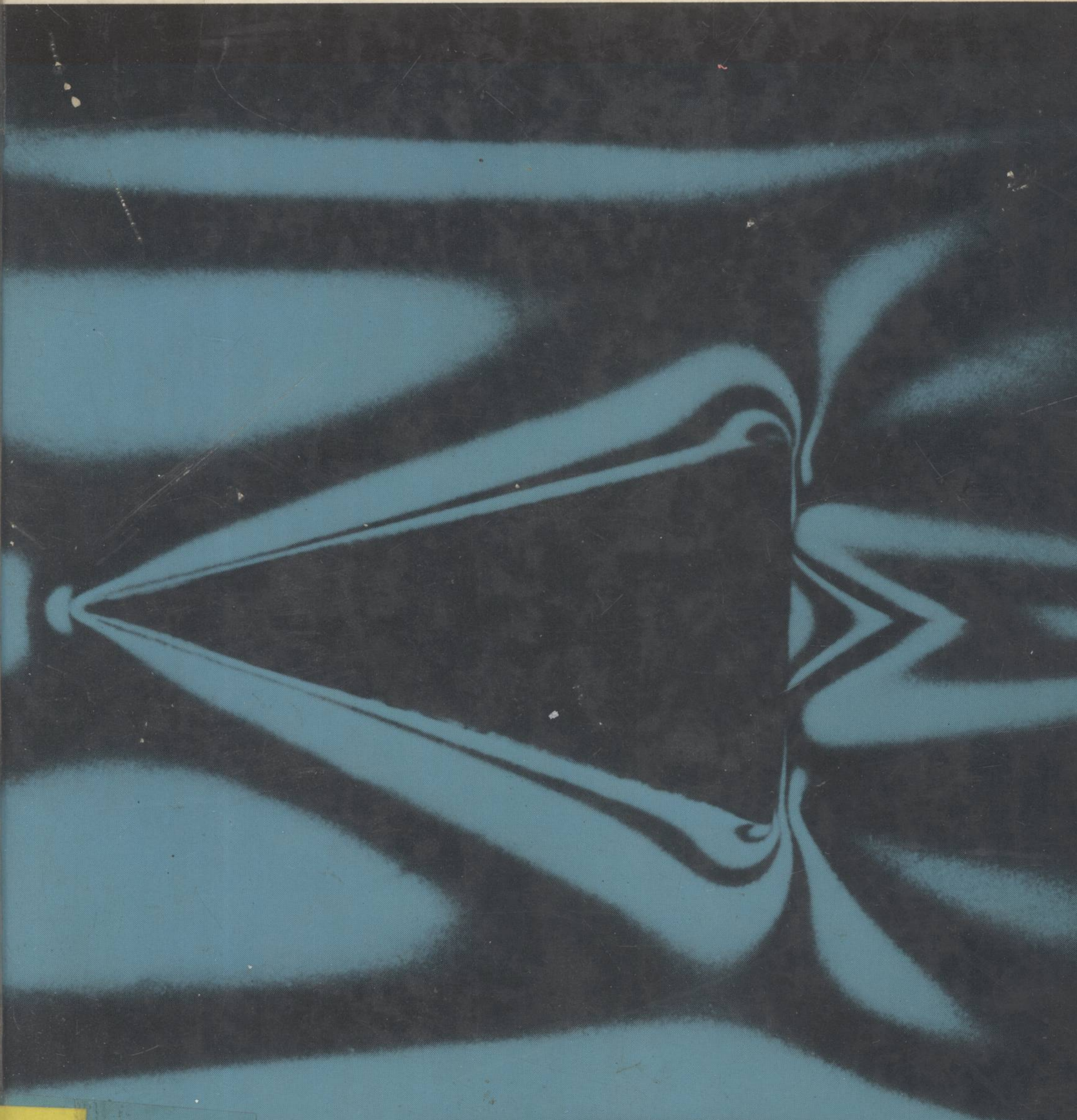
Rheology and non-Newtonian flow



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Rheology and non-Newtonian flow

To Karl Weissenberg 1893–1976

Preface

This work is intended as a text for engineers. It is pedagogical in nature and designed to provide examples of the theory and fundamental problems of non-Newtonian fluid mechanics rather than a review of the published literature, a research monograph, or a text of engineering processes.

I have watched with interest the gradual infiltration of non-Newtonian fluid dynamic studies into university engineering degree courses, and also the growth of the use of the tensor calculus in engineering science, both over the last fifteen years or so. It would seem that the time is ripe now for a text woven from these two threads.

There are several books available at present on the theme of the industrial aspects of non-Newtonian flow and also several more on the rheological properties of matter. (The reader may like to consult *Rheology*, Volumes I–V, edited by Eirich and published by the Academic Press for the latter.) The total number of books available on rheology and non-Newtonian flow is still relatively small, and the present work complements those already available and does not overlap them to any extent.

Earlier this century, the diverging subjects of theoretical hydrodynamics and hydraulics achieved some unification under the stimulus of the boundary layer concept due to Prandtl (1904). This provided a common meeting ground for those with either theoretical or practical engineering inclinations and thus established one of the several areas of growth of engineering science. Little has been known or understood about the so-called non-Newtonian fluids until much more recently; these fluids were termed anomalous and the few early studies were well outside the mainstream of fluid dynamics experiments. The remarkable accord established between theoretical and experimental research was achieved only with rheologically simple fluids such as air or water.

The theoretical investigations of non-Newtonian fluid dynamics are now on a sufficiently sound basis and also of an extent where it may be considered that it can take its place as an established subject and be as academically sound as the more classical studies in fluid dynamics. However, it cannot be said that the experimental side is quite so firmly based. Thus the same accord has yet to be reached in non-Newtonian fluid dynamics between theory and experiment as exists in the classical theory. It is pertinent to mention that the experimental problems are extensive and often present difficulties not present with Newtonian fluids. Reference may be made to the celebrated difficulty in determining extra-normal stresses, and also the viscosity–strain rate relations in general fluid motions. There are other unresolved problems in addition to these two; for example, the nature of the influence of flow on the relaxation spectra of various fluids and also the nature of strain and time-dependent fluids.

In essence, the work described in this book rests on the modelling of some prototype fluids which exhibit some, if not all, characteristics observed in real non-Newtonian fluids. It was said of classical theoretical hydrodynamics and practical hydraulics, perhaps somewhat facetiously, that the former comprised theoretical predictions which could not be observed, whereas the latter comprised observations which could not be predicted. However, a degree of unification has been achieved, as mentioned previously. The theoretical predictions provided archetypal solutions to broad categories of problems, while the practical observations provided the authority to reduce the complexity of theoretical treatments by orders-of-magnitude arguments. This pattern may also be confidently expected to be fruitful in the field of non-Newtonian flow.

In the spirit of the above argument, the first three chapters of this book refer to experimental techniques and direct attention to some means of gaining physical experience of non-Newtonian properties. Later chapters give examples of different types of solutions to a range of fundamental flow problems. It is not intended to imply that the methods of solution described are the only ones available for these particular problems; they are only illustrations of methods.

The physical phenomena in the later chapters are often closely linked – for example, the subject of stability in Chapter 10 is related to Chapter 7 on shear wave propagation; also the results of Chapter 7 should be related to those of Chapter 6 on transient flow via an integral transform procedure and its inverse. Stability of flow is of course also fundamental to all the flow fields treated here. Again Chapter 11 on swirling flow about bodies of revolution is related to Chapter 9 on boundary layers.

Sufficient references have been included for the interested reader to make an entry into the published literature in this field of study. These references are not intended to be exhaustive; the British Society of Rheology publishes a quarterly literature review in *Rheology Abstracts* (Pergamon Press Ltd, Oxford, UK), this provides a comprehensive survey of the world literature pertaining to the subject matter of this book.

At this stage in the development of non-Newtonian fluid dynamics, it was felt more appropriate to concentrate on the physics of flow rather than numerical methods. For the same reason, no reference has been made to variational methods, which do not clarify the physics of the problems in hand, but rather, enable rapid solutions to be obtained to them which do not depend to any extent on the pattern of the flow field.

Acknowledgments

This text is the manifestation of a suggestion made by Professor D. Dowson of the Institute of Tribology, Department of Mechanical Engineering, University of Leeds, United Kingdom. The title is also his suggestion and it provided the seed from which the work finally took shape, it was therefore of seminal influence.

I am grateful to the staff of Longman Group Limited for making invaluable comments and suggestions during the preparation of the material for the press. I sincerely hope that the final result will be a fitting reward for their unflagging interest.

I must express my thankfulness to my wife Marian and children, Jennifer, Andrea, James and Julia.

The Universities of Wales, St. Andrews, Bradford and Nairobi provided me with facilities for undertaking studies contributing to the material presented in this work.

John Harris
Wales, December 1976

Introduction

An historical note on rheology

In the study of physical events it is necessary to define different physical quantities which may be assigned unambiguous numerical values and which may be related to each other. Rheology may be defined as the study of relationships between "stress" and corresponding "strain" in a non-rigid substance.

Hooke's law (1676) is probably the first recognisable rheological law known and it states in effect that deformation is proportional to the applied force. Boyle's law (1662) is of a different character since it relates pressure to *total* volume (of a gas) and is therefore a thermodynamic equation of state. Newton (1686) considered the behaviour of an imaginary fluid taken to fill all space, in which the resistance to motion was proportional to the rate of shear (Newton's viscous law). Deviations from this latter simple state of affairs has provided the terminology "non-Newtonian"; this is the essence of this book.

It was not until about two hundred years later than the advent of the above laws that the first signs of any complication appeared when James Clerk Maxwell (1867) suggested that all substances, including gases, should possess both viscosity and elasticity in shear, and he proceeded to calculate the rigidity modulus of dry air.

On the experimental side Poiseuille (1847), although primarily concerned with the flow of blood (which we now know is non-Newtonian), through capillary tubes, successfully derived from his experiments the relationship between pressure and volumetric flow rate for a Newtonian fluid. Barus (1893) also used a capillary tube, through which he extruded marine glue, and published a paper which was apparently overlooked until more recent years. The important observation in this work was that the glue exhibited a *time-delayed partial recovery of deformation*. As far as is known, this is the first recorded direct observation of non-Newtonian behaviour and is especially important as shear elasticity has now become a major factor in the treatment of non-Newtonian flow.

In the late 1920s and 1930s it became realised that there exists large variations in type of non-Newtonian behaviour. This realisation brought into being the science of Rheology, its birth being largely associated with the names of E. C. Bingham, W. Weissenberg, M. Reiner, G. W. Scott-Blair and others.

Later came the knowledge that interfaces show similar two-dimensional properties to those found in three dimensions in shear, also that both elastic and viscous volumetric effects may be found. The present work is only concerned with shear effects and the fluids throughout are considered to closely approximate the ideal incompressible case.

Tensor notation

The tensor notation for summation followed here is the usual convention of ignoring the summation sign so that:

$$\sum_{i=1}^{i=N} a_i x^i = a_i x^i \quad (i = 1, 2, \ldots N)$$

in general co-ordinates, or

$$\sum_{i=1}^{i=N} a_i x_i = a_i x_i \quad (i = 1, 2, \ldots N)$$

in cartesian co-ordinates.

Also considering a co-ordinate system $x^i = (x^1, x^2, x^3)$ then a new co-ordinate system $\bar{x}^k = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ may be defined by a co-ordinate transformation,

$$\bar{x}^i = \phi^i(x^1, x^2, x^3)$$

Differentiation of this yields,

$$\mathrm{d}\bar{x}^i = \sum_{r=1}^{r=N} \frac{\partial \phi^i}{\partial x^r} \mathrm{d}x^r$$

or with the summation convention,

$$\mathrm{d}\bar{x}^i = \frac{\partial \phi^i}{\partial x^r} \mathrm{d}x^r$$

i.e.

$$\mathrm{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^r} \mathrm{d}x^r$$

A guide to tensor calculus is given in Appendix A IV and there are several good texts available on the subject.

Terminology

The current English usage in rheology has become formalised to a certain extent in a recently issued British Standard, namely, BS5168: 1975. The terminology defined in this standard coincides with that adopted in this book with few exceptions.

The term "viscoelastic" is widely used in the engineering literature in connection with elastic liquids and although this deviates from the more correct terminology of the above British Standard, where elastoviscous (or less euphonically, elasticoviscous) is recommended; it is sanctioned by common current usage and is therefore adopted in this text.

"Viscoelastic" is a term more exactly applicable to a solid-like substance, but no problems arise in practice by using one term for all materials exhibiting both viscosity and elasticity, since the term is normally used as an adjective.

The term "thixotropy" is used in this text in denoting any change in viscosity, from whatever cause, which is apparent in an intrinsic (convected, rotating and deforming with the fluid) co-ordinates system as defined in Chapter 4. The recovery on cessation of shearing motion being full, partial (the more usually observed case) or zero. This rather wide definition is pending further clarification of the nature of thixotropy. The definition given in the British Standard is cast in a looser and less exact form than this.

A further term which requires amplification as a possible source of confusion is that of "viscosity". This term is used here to simply denote the shear stress divided by the *corresponding* shear rate (more exactly, the extra stress tensor divided by twice the corresponding strain rate tensor) irrespective of whether the flow is Newtonian or non-Newtonian. This also applies to elongational flow and is therefore consistent. The prefix "apparent" is not used.

Solids and liquids

In the study of the rheological properties of matter there appears to be no sharp distinction between substances that may be classified as solids or liquids, instead one is faced with an apparently gradual gradation of behaviour between the extremes of a hard brittle solid to that of a mobile liquid, ranging through substances termed "semi-solids".

However, this text is concerned with fluids and solid-like behaviour is to be excluded from consideration. A search must therefore be made for a suitable criterion for the substances falling under consideration; it is this: If the substance exhibits no reference configuration of permanent significance within the time-scale of observation, then the substance is classed as a fluid and may properly come within the scope of this text.

Shear rate, strain rate and deformation rate

These three terms are sometimes not distinguished between, but in this text the following convention will be adopted: Shear rate will be defined as the simple velocity gradient following the practice in many engineering text books. This is actually twice the gross deformation rate because it contains a rigid body rotation.

In many works the overall deformation rate is called the strain rate, and this text follows this convention. However, if the substance contains a deforming network and therefore a statistically quantifiable strain, the overall deformation rate *may* not be the same as the time rate of change of the deformation of the network, or elastic strain.

No reference is made to the deformation or strain of any internal network of the substance, hence no confusion can arise from the use of the term "strain rate" here.

Rheometry and rheogoniometry

In principle, the isothermal and isobaric flow properties of stable Newtonian fluids may be adequately characterised by one pair of experimental observations. This has led to widespread adoption of simple commercial viscometers and easy interpretation of results (unless great refinement is required, in which case there are considerable problems even with Newtonian fluids). For a Newtonian fluid the appropriate flow property is the *viscosity*, i.e. the ratio between shear stress and the corresponding shear rate. This may be obtained for example from a single measurement of volume flux (Q) and pressure gradient ($\Delta p/L$) in the ubiquitous capillary viscometer using the well-known Poiseuille formula; the viscosity is,

$$\mu = \frac{\pi \Delta p a^4}{8QL}.$$

In contrast, the characterisation of a non-Newtonian fluid requires many observations, possibly of several different types, and hence the problem is vastly more complicated than for a Newtonian fluid. The corresponding experimental facilities are therefore generally more complex and costly. The instruments are often of individual design, although there are several commercial designs based upon both the *Poiseuille* and *Couette* type of flow patterns (see Chapter 2). Such instruments are called Rheometers, or Rheogonimeters, since sometimes the stresses in more than one direction in space are required to be found.

Non-Newtonian fluid statics

It is quite common in standard fluid mechanics texts to start with a consideration of hydrostatics. In the case of fluids with no yield stress, solutions in hydrostatics will also apply to non-Newtonian fluids provided the fluid has had sufficient time for its internal structure to relax to a terminal condition.

The statics of other cases has not been treated up to the present time. Slip-line field theory may be of value in the cases where there is a finite yield stress, but applications have not yet been made and no publications exist in rheological or fluid mechanics journals which could be embodied into a non-Newtonian fluid mechanics text.

Notation

1 Symbols: general comments

1.1 x, y, z generally refers to an orthogonal cartesian co-ordinate system.
r, θ, z generally refers to a cylindrical co-ordinate system.
r, α, ϕ generally refers to a spherical co-ordinate system.
y^1, y^2, y^3 also refers to an orthogonal cartesian co-ordinate system.
x^1, x^2, x^3 refers to a general curvilinear co-ordinate system.

1.2 Unbracketed lower case subscripts or superscripts denote tensor components. Bracketed and upper case subscripts imply non-tensor components.

1.3 The symbol Δ or d denotes "a small increment of ...".

1.4 $\times$ denotes the vector cross-product.

1.5 ∇ denotes the Laplacian operator

$$\left(=i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\text{ in cartesian co-ordinates}\right).$$

1.6 Dimensionless groups are signified by N with a qualifying upper case subscript. For example N_R denotes Reynold's number. The Reynolds number is, in most cases, further qualified by an additional subscript.

1.7 Four different types of time derivative are used in this text:

$\frac{\partial}{\partial t}$ denotes the time derivative at a fixed point within the flow fluid.

$\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the Eulerian time derivative tracking the path of a fluid particle.

$\frac{\mathscr{D}}{\mathscr{D}t}$ denotes the Jaumann time derivative tracking the path and rotation of a fluid particle. (Equation (4.81) refers.)

$\frac{\eth}{\eth t}$ denotes the intrinsic time derivative tracking the path, rotation and deformation of a fluid particle. (Equations (4.50) and (4.51) refer.)

The following is a list of notation used in this text. In addition some symbols are defined locally within the text.

2 Definition of notation

2.1 Latin lower case symbols

a tube (or sphere) radius.
a_i acceleration vector.
c_0 initial concentration of disperse phase.
$c(t)$ concentration of disperse phase at current time (t).
e $=2{\cdot}718\ldots.$
e_{ik}, e^{ik} covariant and contravariant strain rate tensor respectively.
g_{ik}, g^{ik} covariant and contravariant metric tensor respectively.
$g(s)$ Laplace transformed distribution function, defined by equation (6.60).
h height of the channel.
k_1, k_2 constants in the power-series equation (7.48).
l light path length.
m mass.
n power-law flow index.
$n_I n_{II}$ principal values of the refractive index.
p_{ik}, p^{ik} covariant and contravariant total stress tensor respectively.
p'_{ik}, p'^{ik} covariant and contravariant extra stress tensor respectively.
s Laplace transform variable.
s^+ complex axial displacement in tube flow.
s', s'' real and imaginary parts respectively of s^+.
t current time.
t', t'' non-current time.
u_i, u^k covariant and contravariant velocity tensor respectively.
$u_{(m)}$ mean velocity.
$u_{(s)}$ effective slip velocity.
u, v, w velocity components.

2.2 Latin upper case symbols

A instrument constant, Table 2.1: also a constant.
B instrument constant, Table 2.1: also a constant.
C couple acting on a disc or a sphere: also a constant.
C_N Nth coefficient.
C_f friction coefficient, defined by equation (9.168).
C_D drag coefficient.
D tube diameter.
D_r rotational diffusion coefficient.
F total thrust on a cone (or plate).
F_B buoyancy force.
F_D drag force on a sphere.
F^i contravariant body force vector.

G rigidity modulus.
$H(t-a)$ Heaviside's step-function.
$I_{(n)}$ nth moment of the time-dependent moment of the thixotropic relaxation distribution spectrum, defined by equation (4.149).
$J_N(x)$ Bessel's function of the first kind, order N.
K power-law coefficient.
$K_{(f)}$ constant.
L characteristic length.
L_{ij} defined by equation (4.85).
$M(\bar{\psi})$ dimensionless viscoelastic relaxation spectrum, defined by equation (6.25).
$M(t-t')$ rheo-optic memory function, defined by equation (3.56).
$\mathcal{M}(t-t')$ viscoelastic memory function, defined by equation (6.22).
$\mathcal{M}_T(t-t')$ thixotropic memory function, defined by equation (1.21).
N Avogadro's number.
$N(\lambda)$ viscoelastic relaxation spectrum.
$N_{(r)}$ local Reynold's number on a rotating disc, defined by equation (11.7).
N_C couple coefficient, defined by equation (11.12).
N_R Reynold's number defined by equation (11.13).
N_Q dimensionless flow number defined by equation (11.19).
N'_R, N''_R real and imaginary parts respectively of the complex Reynold's number.
N_E the ratio N'_R/N''_R.
N_{Rl} length Reynold's number.
N_{RN} Reynolds number relating to a power-law fluid, defined by equation (9.178).
N_{RTx} turbulent length Reynold's number defined by equation (9.189).
N_{RL} length Reynold's number defined by equation (9.132).
N_λ dimensionless number $(=\sqrt{(\lambda\nu/a^2)})$; also defined by equation (9.136).
$O(\ )$ order of magnitude of . . . .
$P(t)$ the negative of the pressure gradient (function of time).
Q volumetric flow rate.
R cone radius; also the gas constant.
$R(r)$ radial space function.
$R(\text{II})$ thixotropic relaxation spectrum.
$R_{(n)}$ nth moment of the thixotropic relaxation spectrum.
R_{ij} defined by equation (4.86).
$S(\chi)$ dimensionless space function, defined by equation (6.34).
$T(\xi)$ dimensionless time function, defined by equation (6.48).
U mainstream velocity; also characteristic velocity.
$X(x)$ spacial distribution function of the stream function (ψ) in the x-direction.
$Y(y)$ spacial distribution function of the stream function (ψ) in the y-direction.
$Y_N(x)$ Bessel's function of the second kind, order N.

2.3 Greek lower case symbols

α Maxwell birefringence constant; also a constant; also a wave number, defined by equation (7.27); also dimensionless momentum thickness, defined by equation (9.25).
α^+ dimensionless complex displacement.
α', α'' real and imaginary parts respectively of α^+.
α_D dimensionless displacement thickness, defined by equation (9.28).
α_E dimensionless energy thickness, defined by equation (9.210).
α_{ij} typical covariant tensor of rank two in an intrinsic co-ordinate system.
β_N relaxation time of the rate process of the *N*th. flow unit in the Eyring viscosity equation (1.11).
β^{rs} contravariant birebringence tensor.
$\tilde{\beta}^{rs}$ extra-birefringence tensor, defined by equation (3.55).
$\dot{\gamma}$ shear rate.
$\dot{\gamma}_{\mathrm{I}}$ maximum principal strain rate.
$\dot{\gamma}_0$ shear rate amplitude in oscillatory motion.
$\gamma_{(n)}$ integral defined by equation (9.211).
γ_{ij} covariant metric tensor of the intrinsic (ζ^i) co-ordinate system.
δ_{ij} Kronecker delta; $\delta_{ij} = 1$ $(i = j)$, $\delta_{ij} = 0$ $(i \neq j)$.
δ phase difference of emerging rays: also boundary layer thickness.
ϵ perturbation parameter.
ζ^i the convected co-ordinate system.
η dimensionless length co-ordinates: defined by equation (7.12).
η' Lamé elastic coefficient.
η_a, η_2 material constants.
θ angle between stream line and the maximum principle stress direction: also dimensionless velocity deficit $(= 1 - \phi)$.
λ viscoelastic relaxation time: also wave length of light.
λ_E time constant.
λ_v shear rate time constant.
λ_1, λ_2 time constants.
μ steady shear viscosity.
μ^+ complex viscosity.
μ', μ'' real and imaginary parts respectively of the complex viscosity.
μ_p plastic viscosity.
μ_∞ high shear rate viscosity.
μ_0 initial viscosity.
$[\mu]$ intrinsic viscosity.
$\hat{\mu}$ Lamé viscosity.
μ_{E} elongational viscosity.
ν kinematic viscosity.
ν_0, ν_1, ν_2 material constants.
ξ dimensionless time; defined by equations (6.24).
π $= 3{\cdot}14159$.
π_{ij} total covariant stress tensor in the ζ^k co-ordinate system.

ρ fluid density.
σ dimensionless co-ordinate.
$\sigma_1, \sigma_2, \sigma_3$ first, second and third normal stress differences respectively.
σ'_1, σ'_2 material constants: defined by equations (4.118) and (4.119).
τ shear stress.
τ_0 shear stress on a solid boundary.
$\tau_{xx}, \tau_{yy}, \tau_{zz}$ total normal stress in the x, y, z directions respectively.
$\tau'_{xx}, \tau'_{yy}, \tau'_{zz}$ extra normal stress in the x, y, z directions respectively.
τ_{I}, τ_{II} maximum and minimum principal stresses respectively.
τ_y yield stress.
ϕ spacial angle between principal stress and principal strain rate: also birefringence orientation angle between n_I and stream line: also dimensionless velocity.
ϕ^+ complex velocity.
ϕ', ϕ'' real and imaginary parts respectively of ϕ^+.
ϕ_0 Newtonian solution for the shear wave velocity: equation (7.55).
ϕ_1 first perturbation velocity from Newtonian shear wave; equation (7.55).
ψ stream function: viscoelastic memory function.
$\bar{\psi}$ dimensionless relaxation time: defined by equations (6.24).
ω angular date of rotation.
χ extinction angle; also dimensionless co-ordinate defined by equation (6.24).

2.4 Greek upper case symbols

Γ constant: also harmonic function satisfying Laplace's equation.
Δ dilation or volumetric strain; defined by equation (4.28).
Θ angular amplitude of driven platen.
Λ^i unit orientation vector.
Π thixotropic relaxation time; also birefringence relaxation time.
Σ summation sign.
Φ angular amplitude of suspended platen.
Φ^+ complex angular amplitude.
Φ', Φ'' real and imaginary parts respectively of Φ^+.
Ψ memory function; defined by equation (4.123).
Ω angular rate of rotation of a disc or sphere.