ELBS

Complex Numbers

Walter Ledermann

LOW-PRICED EDITION

COMPLEX NUMBERS

BY

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The English Language Book Society
and
Routledge & Kegan Paul

First published 1960
in Great Britain by
Routledge & Kegan Paul Limited
39 Store Street
London WC1E 7DD and
Broadway House, Newtown Road
Henley-on-Thames, Oxon RG9 1EN

Printed in Great Britain by
Whitstable Litho Limited, Whitstable, Kent

Second impression (with some corrections) 1962
Reprinted 1964, 1965, 1967, 1971 and 1976

ELBS edition first published 1971
ELBS edition reprinted 1977

ISBN 0 7100 8634 2

COMPLEX NUMBERS

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Preface

THE purpose of this book is to present a straightforward introduction to complex numbers and their properties. Complex numbers, like other kinds of numbers, are essentially objects with which to perform calculations according to certain rules, and when this principle is borne in mind, the nature of complex numbers is no more mysterious than that of the more familiar types of numbers. This formal approach has recently been recommended in a Report† prepared for the Mathematical Association. We believe that it has distinct advantages in teaching and that it is more in line with modern algebraical ideas than the alternative geometrical or kinematical definitions of $\sqrt{-1}$ that used to be proposed.

On the other hand, an elementary textbook is clearly not the place to enter into a full discussion of such questions as logical consistency, which would have to be included in a rigorous axiomatic treatment. However, the steps that had to be omitted (with due warning) can easily be filled in by the methods of abstract algebra, which do not conflict with the 'naïve' attitude adopted here.

I should like to thank my friend and colleague Dr. J. A. Green for a number of valuable suggestions, especially in connection with the chapter on convergence, which is a sequel to his volume *Sequences and Series* in this Library.

WALTER LEDERMANN

† *The Teaching of Algebra in Sixth Forms*, Chapter 3. (G. Bell & Sons, Ltd., London, 1957.)

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CHAPTER ONE

Algebraic Theory of Complex Numbers

1. NUMBER SYSTEMS

Before defining complex numbers let us briefly review the more familiar types of numbers and let us examine why there are different kinds of numbers.

The most primitive type of number is the set of *natural numbers* 1, 2, 3, . . ., which the child learns for counting objects. Arithmetic, the science of numbers, is based on the fact that numbers can be added and multiplied, subject to certain rules, to which we shall presently return in more detail. It is the existence of these two laws of composition and their mutual relation that we shall regard as the typical feature of all numbers and that will serve us as a guide for introducing new systems of numbers for various purposes.

Let us recall how in the school curriculum we proceed from the natural numbers to more elaborate systems. The attempt to make subtraction always possible, that is to solve the equation $a+x=b$ for x when a and b are given, leads to the introduction of zero (one of the great achievements of the human mind!) and of the negative numbers. We now have the set of all *integers* (whole numbers) . . . -3, -2, -1, 0, 1, 2, 3, . . . Next, when we wish to carry out division, we have to solve equations of the form $ax=b$, where a and b are given integers and a is non-zero. In order to make the solution possible in all cases it is necessary to introduce the *rational numbers* (fractions). These numbers are denoted by symbols b/a, where a and b are integers and a is non-zero.

When this stage has been reached, the four rules of arithmetic, that is addition, subtraction, multiplication and division apply without restriction, always excepting division by zero. These basic operations are governed by the following general laws, which are of fundamental importance in mathematics.

I. $a+b=b+a$ (**commutative law of addition**).
II. $(a+b)+c=a+(b+c)$ (**associative law of addition**).
III. $a+x=b$ has a unique solution, written $x=b-a$ (**law of subtraction**).
IV. $ab=ba$ (**commutative law of multiplication**).
V. $(ab)c=a(bc)$ (**associative law of multiplication**).
VI. $ax=b$ $(a\neq 0)$ has a unique solution $x=b/a$ (**law of division**).
VII. $(a+b)c=ac+bc$ (**distributive law**).

Most of these laws, perhaps in a different guise, are so familiar to the reader that he might be unaware of their existence. Thus the associative law of addition implies that a column of figures can be added by starting either from the top or from the bottom. Again, the distributive law is more popularly known as the principle of multiplying out brackets.

The rational numbers are adequate for dealing with the more elementary questions of arithmetic, but their deficiency becomes apparent when we consider such problems as extracting square roots. For example, it can be shown that $\sqrt{2}$ cannot be expressed in the form m/n, where m and n are integers, *i.e.* there are no integers m, n $(\neq 0)$ such that $m^2=2n^2$. Again, when we pass from algebra to analysis, where limits of sequences play a fundamental part, we find that the limit of a sequence of rational numbers is not necessarily a rational number.† The situation may be described by using a single co-ordinate axis

† See J. A. Green, *Sequences and Series*, in this series, p. 7.

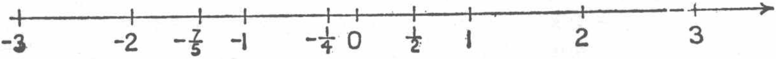

Figure 1

on which in the first place we mark all the integers in a certain scale. Then we imagine all the rational numbers inserted, e.g. $-7/5$, $-1/4$, $1/2$, . . . But even when this has been done, there will be many points on the line against which no number has been entered. For instance when we lay down a segment of length $\sqrt{2}$ (the diagonal of a square of unit sides) by placing one end at 0, the other end-point falls on a point of the scale which has as yet no number attached to it. On the other hand, we intuitively accept the fact that every segment ought to have a length which is measured by some 'number'. In other words, we postulate that every point on the axis possesses a co-ordinate which is a definite number, positive if the point is on the right of 0 and negative if it is on the left of 0. This number need not be a rational number. The set of numbers which in this way fill the whole line, is called the set of *real* numbers; they comprise the familiar rational numbers, the remaining real numbers being called *irrational*, such as $\sqrt{2}$, e, π, log 2, etc. (Of course, the word irrational means that the number is not the *ratio* of two integers and has nothing to do with the idea that something irrational is beyond the realm of reason.) Alternatively, the real numbers may be described as the set of all decimal fractions. A terminating or a recurrent decimal fraction corresponds to a rational number, whilst the other fractions represent irrational numbers.

From the way in which real numbers are depicted on a line it is clear that there exists an *order relation* among them, that is any two real numbers a and b satisfy either $a=b$ or $a<b$ or $a>b$. This is indeed an important property when we wish to use numbers for measuring. But in the present algebraical context we are much more concerned with the fact that real numbers, like rational numbers, can be added and multiplied and that they obey the laws I to VII listed on p. 2. We take the view that the existence of the two modes of composition with their laws makes numbers deserve their name. Numbers are essentially things to be computed, and any other properties, however useful for certain purposes, are not part of the definition of number. One of these secondary properties is the fact that real numbers can be classified into positive and negative numbers together with the usual deductions from it, such as 'the product of two negative numbers is positive'.

For a long time it was held that arithmetic had reached saturation with the introduction of the complete set of real numbers. Indeed, there was no obvious geometrical or technical problem that called for the creation of new numbers. Yet, one of the simplest algebraical questions remains in an unsatisfactory state when only real numbers are available. For we should then be forced to admit that some quadratic equations have solutions whilst others have none. On the other hand, it is easy to see that all quadratic equations would have solutions if only we could solve the special equation

$$x^2+1=0, \tag{1.1}$$

for this would assign a meaning to $\sqrt{-1}$ and hence to $\sqrt{-a}$, where a is any positive number. Indeed, we could simply put $\sqrt{-a}=\sqrt{-1}\sqrt{a}$. Now it is obvious that (1.1) cannot have a real solution, since if x is real, x^2 is never negative and cannot therefore be equal to -1. So in order to make (1.1) soluble we have to introduce a new type of number, for which the rule 'the square of any number is positive' certainly does not hold. But this rule, or indeed anything else concerning positiveness and negativeness is not a consequence of the seven fundamental laws listed on p. 2, and it is therefore quite conceivable that these laws can be satisfied by symbols or numbers to which the terms positive and negative do not apply.

We now formally introduce a symbol i which we treat in the same way as an indeterminate x in algebra, except that i has the additional property that

$$i^2 = -1. \tag{1.2}$$

More precisely, we (tentatively) postulate that when i is adjoined to the existing real numbers, addition and multiplication in the enlarged system will still obey the seven fundamental laws despite the bizarre stipulation (1.2). On this assumption, we deduce from (1.2) that

$$i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1, \ldots \tag{1.3}$$

Thus a polynomial in i, that is an expression of the form $a_0 + a_1 i + a_2 i^2 + a_3 i^3 + a_4 i^4 + \ldots + a_n i^n$, where the co-efficients $a_0, a_1, \ldots, a_n$ are real, reduces to the simple form $a+ib$, where $a = a_0 - a_2 + a_4 - \ldots$ and $b = a_1 - a_3 + a_5 - \ldots$ are real numbers. A symbol of the form

$$\alpha = a + ib \text{ or } a + bi$$

where a and b are real, will be called a *complex number.* The algebraical and other properties of these numbers, as we shall feel justified in calling them, will be studied in the remainder of this book.

2. THE ALGEBRAIC THEORY

The first prerequisite for a set of objects to qualify as numbers is that they should be capable of being added and multiplied. The natural way to define addition is to put

$$(a+ib)+(c+id) = (a+c)+i(b+d), \tag{1.4}$$

collecting terms with i and terms without i. For example, $(3+2i)+(5+6i) = 8+8i$, $(-1+4i)+(2+(-7)i) = 1+(-3)i$. As regards multiplication, we obtain by formal multiplica tion

$$(a+ib)(c+id) = ac + adi + bci + bdi^2,$$

whence by (1.2),

$$(a+ib)(c+id) = (ac-bd)+i(ad+bc). \tag{1.5}$$

The definitions (1.4) and (1.5) constitute the basis for an algebraical treatment of complex numbers. Although these definitions appear to be quite natural or even obvious, they can be accepted only if they are compatible with the seven fundamental laws. This is indeed the case, but the verification of this fact is somewhat tedious and we ask the reader to take it on trust.

A complex number $a+ib$ is completely given when the real numbers a and b are known. The numbers $a+ib$ and $c+id$ are equal if and only if simultaneously $a=c$ and $b=d$. Thus an equation involving complex numbers is equivalent to two equations between real numbers.

It is possible to think of a complex number as an *ordered pair* of real numbers (a, b), and the formulae (1·4) and (1·5) then correspond to rules for adding and multiplying such pairs. Thus

$$(a, b)+(c, d)=(a+c, b+d)$$
$$(a, b)(c, d)=(ac-bd, ad+bc).$$

However, we prefer to regard a complex number as a single mathematical entity and, whenever possible we use a single letter to denote a complex number, thus $\alpha=a+ib$.

The real numbers a and b are called the *real part* and the *imaginary part* of α respectively, and we write

$$a=\mathscr{R}\alpha, \quad b=\mathscr{I}\alpha.$$

Note that the imaginary part of α is in fact a real number. When $\mathscr{I}\alpha=0$, the complex number α reduces to $a+i0$, and this symbol behaves in every way like the real number a. In this case the rules for addition and multiplication reduce to

$$(a+i0)+(c+i0)=a+c+i0$$
$$(a+i0)(c+i0)=ac+i0.$$

We shall therefore simply write a for $a+i0$ and we accordingly regard the real numbers as special cases of complex numbers, namely those complex numbers whose imaginary parts are zero. Note, in particular, that the multiplication of a complex number by a real number follows the simple rule.

$$a(c+id)=(c+id)a=ac+iad.$$

The complex zero and the complex unit are the same as the real 0 and 1. A complex number of the form ib, whose real part is zero is called a *purely imaginary* number. There is no need to comment on the contraction of notation whereby $a+(-b)i$ is written as $a-ib$. Subtraction is evidently given by the formula

$$(a+ib)-(c+id)=(a-c)+i(b-d).$$

We defer the discussion of division until we have introduced a few more useful concepts and formulae.

With every complex number $\alpha=a+ib$ we associate the *conjugate* complex number $\bar{\alpha}=a-ib$. Thus $\bar{\alpha}=\alpha$ means that α is real, that is $b=0$; $\bar{\alpha}=-\alpha$ holds if and only if α is purely imaginary. The passage from α to $\bar{\alpha}$ consists merely in replacing i by $-i$. It should be noted that every significant algebraical statement about i is also true about $-i$, because both symbols satisfy the defining relation $i^2=(-i)^2=-1$.

It is easy to verify the important rules

$$\overline{\alpha+\beta}=\bar{\alpha}+\bar{\beta} \tag{1.6}$$

$$\overline{\alpha\beta}=\bar{\alpha}\,\bar{\beta} \tag{1.7}$$

For example, (1.7) means explicitly that in the notation of (1.5) $(ac-bd)-i(ad+bc)=(a-ib)(c-id)$. In particular, we have that $\overline{\alpha^2}=(\bar{\alpha})^2$, etc. An interesting result is obtained when we multiply α by $\bar{\alpha}$, thus

$$\alpha\bar{\alpha}=(a+ib)(a-ib)=a^2-(ib)^2=a^2+b^2,$$

which is real and positive, except when $\alpha=0$, in which case it is obviously zero. The non-negative† real number

$$|\alpha|=\sqrt{(a^2+b^2)}=\sqrt{\{(\mathscr{R}\alpha)^2+(\mathscr{I}\alpha)^2\}} \tag{1.8}$$

is called the *modulus* (*absolute value*) of α, and we have that

$$\alpha\bar{\alpha}=|\alpha|^2. \tag{1.9}$$

We remark once more that $|\alpha|=0$ if and only if $\alpha=0$ and that for all complex numbers, other than zero, $|\alpha|>0$. Of course, different complex numbers may have the same

† We adopt the convention that the square root of a non-negative real number always stands for the positive square root.

modulus, for example, conjugate complex numbers always have the same modulus, thus $|\alpha|=|\bar{\alpha}|$. Again, if $\alpha=\cos\theta + i\sin\theta$, where θ is an arbitrary real number, then $|\alpha|=\sqrt{(\cos^2\theta+\sin^2\theta)}=1$. When $\alpha=a$ is real, the definition (1.8) reduces to $\alpha=\sqrt{a^2}$, which is equal to a if $a>0$ and is equal to $-a$ if $a<0$. This agrees with the definition of the modulus $|a|$ of a real number a, which is therefore generalized by (1.8).

Let $\beta=c+id$ be another complex number. We want to consider the modulus of the product $\alpha\beta$. Using (1.7) and (1.9) we find that

$$|\alpha\beta|^2=(\alpha\beta)(\overline{\alpha\beta})=\alpha\beta\overline{\alpha\beta}=\alpha\bar{\alpha}\beta\bar{\beta}=|\alpha|^2|\beta|^2. \qquad (1.10)$$

Since the moduli are never negative the extraction of the square roots introduces no ambiguity, and we arrive at the very simple result that

$$|\alpha\beta|=|\alpha|\,|\beta|. \qquad (1.11)$$

When we wish to translate this result into a statement about real numbers, we work out $|\alpha\beta|$ by substituting in (1.8) the values for $\mathscr{R}(\alpha\beta)$ and $\mathscr{I}(\alpha\beta)$ from (1.5), thus

$$|\alpha\beta|^2=(ac-bd)^2+(ad+bc)^2.$$

Hence (1.10) is equivalent to the interesting identity

$$(ac-bd)^2+(ad+bc)^2=(a^2+b^2)(c^2+d^2),$$

which of course could be readily verified by direct calculation. In particular, when $\alpha=\beta$, the multiplication formula (1.11) becomes $|\alpha^2|=|\alpha|^2$, and by repeating this argument we find that $|\alpha^n|=|\alpha|^n$, where n is any positive integer.

Examples.

$$|i|=|-i|=1,\ |-5|=5.\ |1+i|=\sqrt{(1^2+1^2)}=\sqrt{2}.$$
$$|(1+i)^4|=4.\ |\tan\theta+i|=\sqrt{\{(\tan\theta)^2+1^2\}}=|\sec\theta|.$$

We can now describe a simple solution to the problem of division. Let $\alpha=a+ib$ and $\beta=c+id$ be given complex numbers and suppose that $\alpha\neq0$. It is required to find a number $\xi=x+iy$ such that

$$\alpha\xi=\beta\ (\alpha\neq0) \qquad (1.12)$$

Let us assume for a moment that there exists such a number ξ. Then on multiplying (1.12) by α we find that $\alpha\bar{\alpha}\xi = \bar{\alpha}\beta$, that is

$$(a^2+b^2)(x+iy)=(a-ib)(c+id)=(ac+bd)+i(ad-bc).$$

On comparing real and imaginary parts on both sides we see that

$$x=\frac{ac+bd}{a^2+b^2},\ y=\frac{ad-bc}{a^2+b^2}.$$

Thus if there is a solution at all, it must be given by

$$\xi=\frac{ac+bd}{a^2+b^2}+i\frac{ad-bc}{a^2+b^2}=\frac{1}{|\alpha|^2}\bar{\alpha}\beta. \tag{1.13}$$

Conversely, it is easy to see that (1.13) does in fact satisfy (1.12). Indeed, $\alpha\frac{1}{|\alpha|^2}\bar{\alpha}\beta=\frac{\alpha\bar{\alpha}}{|\alpha|^2}\beta=\beta$. The unique solution, exhibited in (1.13), will be written β/α or $\beta\alpha^{-1}$, or $\alpha^{-1}\beta$. There is no need to memorize the explicit formula for β/α. The argument which shows that there is such a complex number is equivalent to the familiar 'rationalization of denominators' in working with surds. Indeed, it should be borne in mind that, after all, $i=\sqrt{-1}$ is a surd. To simplify a complex fraction we multiply numerator and denominator by the conjugate complex of the denominator, thus

$$\frac{c+id}{a+ib}=\frac{(c+id)(a-ib)}{(a+ib)(a-ib)}=\frac{(ac+bd)+i(ad-bc)}{a^2+b^2}, \tag{1.14}$$

which is equivalent to (1.13). In particular, we note that if $\alpha\neq 0$,

$$\frac{1}{\alpha}=\frac{\bar{\alpha}}{|\alpha|^2}=\frac{a-ib}{a^2+b^2},\ \left|\frac{1}{\alpha}\right|=\frac{1}{|\alpha|}. \tag{1.15}$$

Summarizing, we can say that complex numbers are mathematical objects for which addition, multiplication, subtraction and division are defined in such a way that the seven fundamental laws are satisfied. The real numbers may be regarded as particular cases of complex numbers, so that any general property of complex numbers also holds for real numbers. However, the converse of this statement is not true. For example, there is no simple order relation between complex numbers and the symbol $\alpha<\beta$ is not defined, nor is there any sense in referring to a complex number as being positive or negative. These are attributes of real numbers which cannot be transferred to complex numbers.

The simplest type of problem consists in reducing an expression involving complex numbers to its standard form, that is to the form $x+iy$, where x and y are real. This will be illustrated in the following examples.

Example 1.

$$\frac{(1+2i)^2}{1-i}=\frac{1+4i+4i^2}{1-i}=\frac{-3+4i}{1-i}=\frac{(-3+4i)(1+i)}{(1-i)(1+i)}$$
$$=\frac{-7+i}{2}=-\frac{7}{2}+\frac{1}{2}i.$$

Example 2.

$$\frac{1}{1+i}+\frac{1}{1-2i}=\frac{1-i}{(1+i)(1-i)}+\frac{1+2i}{(1-2i)(1+2i)}$$
$$=\frac{1-i}{2}+\frac{1+2i}{5}=\frac{7}{10}-\frac{1}{10}i.$$

Example 3.

$$1+i+i^2+i^3+i^4+i^5+i^6+i^7=\frac{1-i^8}{1-i}=\frac{1-(i^4)^2}{1-i}=0.$$

Let $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ be a polynominal with real coefficients $a_0, a_1, a_2, \ldots, a_n$. If we substitute for x a complex number α, we obtain the number

$$f(\alpha)=a_0\alpha^n+a_1\alpha^{n-1}+a_2\alpha^{n-2}+\ldots+a_{n-1}\alpha+a_n. \quad (1.16)$$

We now wish to find $f(\bar{\alpha})$. By a repeated application of (1.6) and (1.7) we may do this by placing a bar across each term on the right-hand side of (1.16) and across all the factors of this term. But since the coefficients are real $\bar{a}_0=a_0$,