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# Methods of singular integral equations

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# Preface

The traditional integral equations method was for many years the most powerful method for proving the main existence problems of mathematical physics connected with partial differential equations. Now by means of functional analysis methods and the theory of distributions it has become possible to investigate general partial differential operators. The general theory of partial differential equations has developed in such a way that many contemporary investigations are based on the so-called micro-analysis method arising from the ideas and methods of using pseudodifferential and Fourier integral operators. Thus in the general theory of partial differential equations integral equations no longer plays their traditional role as a tool of investigation.

On the other hand many thin and complete results in the theory of partial differential equations with two independents were obtained by means of the theory of functions of complex variables and the theory of one-dimensional integral equations, especially singular ones.

The main themes of this book are systems of partial differential equations of nonclassical type (including composite type) and the main tool of their investigation, developed by the author, is the method of singular integral equation on multiply bounded domains.

Since systems of partial differential equations of composite type possess at every point of the considered domain both elliptic and hyperbolic properties, the first question arising is how to formulate well-posed problems for them.

In the case of systems with two independent variables, well-posed problems in a bounded domain were formulated in 60 years. In Dzhuraev, 1989 the method of singular functional equations was elaborated and hence the normal solvability theory was formulated and the index of well-posed problems through their coefficients was calculated. Unfortunately this theory was not extensible to higher dimensions, because of many difficulties arising with the theory of singular integral equations in high-dimensional bounded domains and also with the classification of multidimensional partial differential operators.

In two-dimensional bounded multiply connected domains a complete theory of a class of singular integral equations is formulated in this book. This theory allows us to find thin properties of this kind of singular integral equations which differs from the one-dimensional case.

As it is known the Dirichlet problem is the simplest boundary-value problem, which is well posed for scalar elliptic equations, and for general scalar elliptic equations this problem has zero index, i.e. has the Fredholm property. On the other hand there are second-order elliptic equations with complex coefficients for which the Dirichlet problem is ill posed in some domain. This means that for general elliptic systems the Dirichlet problem is not natural - only so-called strongly elliptic systems. Connecting with this is the problem of finding the formulation of well-posed problems for elliptic systems that are not necessarily strongly elliptic. This is important also in the sense that the set of general elliptic systems is more powerful compared with the set strongly elliptic ones. In Chapter 3 we state and study the new boundary-value problems for general elliptic systems which seem to be natural in the sense that they are well posed and for multiply connected domains satisfy the Fredholm property, i.e. have zero index.

In Chapter 4 we state and study initial- and initial-boundary-value problems for nonstationary systems of equations, including systems of composite type. Initial-boundary-value problems reduce to spectral theory for elliptic (not necessarily strongly elliptic) systems of partial differential equations in the case of two-dimensional space variables. All these considerations are based on the theory of singular integral equations over a bounded plane domain.

In concluding Chapter 5 we extend some results of the preceding chapters to the multidimensional case. Unfortunately the singular integral equations method is not used in this chapter, but some connections with generalizations of the Cauchy-Riemann equation allows us to consider some remarkable elliptic multidimensional operators, and to state and study well-posed boundary-value problems for them, as well as initial-boundary-value problems for nonclassical nonstationary systems generated by them.

# Preface to English edition

In this English edition some changes have been made to the original book. In Chapter 1, Sections 11 and 12 are combined as Section 11. In Chapter 2, Section 7 is new. In Chapter 3, Section 2, the formulation of solvability theory of the Dirichlet problem is given in terms of the newly introduced adjoint Dirichlet problem, and in Section 4 the proofs of solvability theory for the $A$-problem are essentially improved. Chapter 4 has been deleted. In Chapter 5, Sections 6–8, 11 and 12 are exchanged for new material expounded in Sections 6–8.

# Notation

$\mathbb{R}^n$ — $n$-dimensional Euclidean space of points $x = (x_1,...,x_n)$, $y = (y_1,...,y_n)$, $\xi = (\xi_1,...,\xi_n)$, $\eta = (\eta_1,...,\eta_n)$

$\mathbb{C}$ — complex plane of the variable $z = x + iy$

$\bar{z} = x - iy$ — complex conjugated point in $\mathbb{C}$

$C^k$ — the space of functions that are continuous and possess derivatives of order $k$

$C^\infty$ — the space of infinitely differentiable functions

$C_0^\infty$ — the space of compactly supported infinitely differentiable functions

$\mathfrak{B}(G)$ — the Bergman space of analytic functions in $G$

$A_n^2$ — the space of polyanalytic functions in $G$

$\mathfrak{A}_p$ — the space of generalized analytic functions

$W_p^\ell$ — the space of Sobolev

$H$ — the Hilbert transformation

$H_j$ — the M. Riesz transformation

$S$ — two-dimensional Hilbert transformation in $\mathbb{C}$

$S_G$ — two-dimensional Hilbert transformation on bounded domain $G$

$K(z, \zeta)$ — the kernel function of the domain

$P(x, D)$ — linear differential operator of the order $m$

$P^*(x, D) \equiv \sum_{|s| \le m} (-1)^{|s|} D^s (\overline{a_{(s)}(x) \cdot})$ — the operator adjoint to $P(x, D)$

# Contents

Preface
Preface to English edition
Notation

**1. Integral representations of functions**

1. Hilbert transformations 1
2. Cauchy's singular integrals and integral equations on the system of curves 9
3. Boundary-value problems of analytical functions 14
4. Special cases of the Riemann-Hilbert problems for analytical functions 19
5. Periods. Neumann's function 26
6. Bergman spaces. The kernel functions of the domain 31
7. Some integral formulae. Generalized derivatives 42
8. The Schwarz problem for the inhomogeneous Cauchy-Riemann equation 45
9. Generalized analytical functions 51
10. Generalized Riemann-Hilbert boundary-value problem 52
11. An integral representation of the functions satisfying different boundary conditions 54

**2. The theory of two-dimensional singular integral equations**

1. Introductory remarks 66
2. The existence of eigenvalues 68
3. The connection of the singular integral equation on a bounded domain on the plane with the general conjugation problem 74
4. The general case 80
5. The investigation of the conjugation boundary-value problems. Main theorems 89
6. The theory of some systems of singular integral equations on bounded domains on the plane 94
7. The investigation of the general conjugation problem 111
8. The kernel operators and their connection with singular integral operators 119
9. The composition of singular integral operators 121
10. On some classes of singular integral equations 128

**3. Boundary value problems for elliptic partial differential equations in the plane**

1. Linear elliptic systems 134
2. Dirichlet and Neumann problems 137
3. Analysis of the integral equations of the Dirichlet and Neumann problems 146
4. The boundary-value problems $A$ and $\bar{A}$ 156
5. Generalization of Bergman's kernel function 175
6. The representation of functions in a bounded multiply connected domain satisfying the homogeneous Dirichlet condition 190
7. Reduction of the Dirichlet problem for high-order elliptic equations to an integral equation 199
8. On boundary-value problems for high-order elliptic systems that are not strongly elliptic 206

**4. Initial and initial-boundary value problems for nonclassical systems of partial differential equations**

1. Initial problems in a half-space 212
2. The simplest initial-boundary-value problems 218
3. The eigenvalues and eigenfunctions of boundary-value problems for elliptic systems in a bounded domain in the plane, with variable coefficients 227
4. Nonstationary problems with variable coefficients 241
5. High-order systems 245

**5. Multidimensional systems**

1. The Cauchy-Green type formulae 259
2. On some multidimensional elliptic systems 264
3. The existence of the eigenvalues and the eigenfunctions 276
4. Initial and initial-boundary-value problems for second-order composite type system 279
5. On boundary-value problems for the Moisil-Theodorescu system 288
6. Well-posed boundary-value problems for second-order elliptic systems that are not necessarily strongly elliptic 294
7. On boundary-value problems for generalized overdetermined vector-field systems of equations 299
8. On some second-order elliptic systems and mixed composite-type systems 302

Bibliography 308

# 1
# Integral representations of functions

This chapter is of an auxiliary character. The necessary information from the theory of one-dimensional singular integral equations, boundary-value problems of analytic and generalized analytic theory of functions is introduced. The classes of the functions satisfying some boundary conditions and an integral representation for them are considered, and these tools are used in later chapters.

## 1. Hilbert transformations

### A. Singular operators in Euclidean space

(1) The simplest example of the singular integral is the Hilbert transformation of the function $f(x)$ given on the real axis

$$Hf = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi - x} \tag{1.1}$$

Since $\int_{-\infty}^{\infty} |x|^{-1}\,dx = \infty$, it follows that integral (1.1) is not absolutely convergent as a usual improper integral, therefore there is not a defined bounded map anywhere in $L_1(-\infty, \infty)$ or $L_\infty(-\infty, \infty)$, but if $f(x)$ has compact support, for example, then (1.1) may be interpreted by means of a special passing to the limit (Cauchy principal value):

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi - x} = \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right\}\frac{f(\xi)\,d\xi}{\xi - x} .$$

One of the remarkable proper;ties of the integral (1.1) is that it determines a bounded operator in the space $L_p(-\infty, \infty)$ with $1 < p < \infty$, i.e. if $f(x) \in L_p(-\infty, \infty)$ then there exists such a constant $A_p$ such that

$$\| H_f \|_p \leq A_p \|f\|_p, \quad 1 < p < \infty \tag{1.2}$$

This inequality, which called the M. Riesz inequality, formed the basis of a multidimensional generalization called the Calderon–Zygmund inequality.

To prove (1.2) the operator (1.1) could be considered on a set of trial functions $C_0^\infty$ which are dense in $L_p$ and then extend continuously to the whole of $L_p$. Though the proof of (1.2) given by M. Riesz is elegant we expound here another no less elegant but more simple proof due to Zygmund (1935). Thus let $f(x) \in C_0^\infty(-\infty, \infty)$ and $p \geq 2$. Consider the Cauchy integral $(z = x + \mathrm{i}\, y)$

$$F(z) = u + \mathrm{i}\, v = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{\xi - x}, \quad \mathrm{Im}\, z > 0.$$

Since $\mathrm{Im}\, F(z)$ is Poisson's integral

$$\mathrm{Im}\, F(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{(\xi - x)^2 + y^2}$$

then $v(x) = \mathrm{Im}\, F(x) = f(x)$. On the other hand,

$$u(x, y) = \mathrm{Re}\, F(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\xi - x}{(\xi - x)^2 + y^2} f(\xi)\, d\xi$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x+\xi) - f(x-\xi)}{\xi} \cdot \frac{x^2}{x^2 + \eta^2}\, d\xi$$

$$\to \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x+\xi) - f(x-\xi)}{\xi} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{\xi - x} = Hf(x).$$

If $\Delta$ is Laplace's operator in variables $x, y$, then evidently from the harmonicity of $u$, $v$ and the Cauchy-Riemann equation we obtain, for $p \geq 2$,

$$\Delta |u|^p = p(p-1) |u|^{p-2} (u_x^2 + u_y^2), \quad \Delta |v|^p = p(p-1) |v|^{p-2} (v + v_y^2),$$

$$\Delta |F|^p = p^2 |F|^{p-2} (u_x^2 + u_y^2).$$

Hence

$$\Delta \left(|F|^p - \frac{p}{p-1} |u|^p\right) = p^2 \left(|F|^{p-2} - |u|^{p-2}\right) (u_x^2 + u_y^2) \geq 0.$$

Let $K_R : |z| \leq R, y > 0$, the upper half of a disk with radius $R$ with centre at the origin. If $C_R$ is the boundary of $K_R$, then by Green's formula

$$\iint_{K_R} \Delta\left(|F|^p - \frac{p}{p-1}|u|^p\right) dx\, dy = -\int_{C_R} \frac{\partial}{\partial y}\left(|F|^p - \frac{p}{p-1}|u|^p\right) ds,$$

i.e.

$$\int_{C_R} \frac{\partial}{\partial y}\left(|F|^p - \frac{p}{p-1}|u|^p\right) ds \leq 0$$

or, since $|F|^p - \frac{p}{p-1}|u|^p \to 0$ when $R \to \infty$, then for $|z| \to \infty$

$$\frac{\partial}{\partial y} \int_{-\infty}^{\infty} \left(|F|^p - \frac{p}{p-1}|u|^p\right) dx \leq 0.$$

On the other hand the integral tends to zero when $y \to \infty$. It follows that for fixed $y > 0$ this integral is not negative:

$$\int_{-\infty}^{\infty} |F(x+iy)|^p\, dx \geq \frac{p}{p-1} \int_{-\infty}^{\infty} |u(x+iy)|^p\, dx.$$

But

$$\left(\int_{-\infty}^{\infty} |F|^p\, dx\right)^{2/p} = \|u^2 + v^2\|_{p/2} \leq \|u\|^2_{p/2} + \|v\|^2_{p/2}.$$

Therefore

$$\left(\frac{p}{p-1}\right)^{2/p} \|u\|^2_{p/2} \leq \|u\|^2_{p/2} + \|v^2\|_{p/2}$$

and

$$\|u^2\|_{p/2} \leq \frac{1}{\left(\frac{p}{p-1}\right)^{2/p} - 1} \|v^2\|_{p/2}$$

i.e.

$$\int_{-\infty}^{\infty} |u(x+iy)|^p\,dx \le \frac{1}{\left(\frac{p}{p-1}\right)^{2/p} - 1} \cdot \int_{-\infty}^{\infty} |v(x\ iy)|^p\,dx.$$

Hence (1.2) follows when $y \to 0$ for $p \ge 2$. If $p < 2$ then taking into account the fact that the adjoint operator $H^*$ to (1.1) coincides with $-H$ and $p' = p(p-1) > 2$, we obtain the inequality (1.2) for the adjoint operator in $L_{p'}$. Whereas for functions given on the real axis there exists a unique Hilbert transformation, for those given on Euclidean space of dimension greater than one there exist uncountably many such transformations.

(2) In order to get a direct generalization of the integral (1.1) we note that it can be written as

$$Hf = \int_{-\infty}^{\infty} \frac{\Omega(\xi - x)}{|\xi - x|} f(\xi)\,d\xi$$

where

$$\Omega(x) = \frac{1}{\pi}\ \text{sign}\ x = \frac{1}{\pi}\frac{x}{|x|}. \qquad (1.1')$$

Then it is clear that for the function $f(x)$ on $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ we can determine (see Riesz 1927; Stein 1970) $n$ such transformations

$$H_j f = \int_{\mathbb{R}^n} \frac{\Omega_j(\xi - x)}{|\xi - x|^n} f(\xi)\,d\xi \qquad (1.3)$$

where

$$\Omega_j(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{x_j}{|x|},\quad j = 1,2,...,n.$$

Integrals (1.3) like integrals (1.1′) are characterized by the property that the $\Omega_j(x)$ are homogeneous odd functions of order zero. This makes it possible to prove an inequality analogous to (1.2) for the integrals (1.3), provided the surface on the unit sphere in $\mathbb{R}^n$ from $\Omega(x)$ is bounded. But if the surface integral on the unit sphere in

$\mathbb{R}^n$ from $\Omega(x)$ is equal to zero we may consider more general integrals in $\mathbb{R}^n$ than (1.3):

$$Kf = \int_{\mathbb{R}^n} \frac{\Omega(\xi - x)}{|\xi - x|^n} f(\xi)\, d\xi \tag{1.4}$$

even in case when $\Omega(x)$ is not odd

The important example of this kind of integral which arises in problems of partial differential equations and will appear further in case $n = 2$ is

$$Sf = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(\zeta) d\xi\, d\eta}{(\zeta - z)^2}, \tag{1.5}$$

where $\mathbb{C}$ is the complex plane $z = x + iy$ ($\zeta = \xi + i\eta$) which identified with $\mathbb{R}^2$. The integral (1.5) has the form (1.4) with $\Omega(z) = (z/|z|)^2$ an even function,and the surface integral from $\Omega(z)$ on the unit sphere in $\mathbb{C}$ is $\int_0^{2\pi} e^{-2i\theta}\, d\theta = 0$.

If the surface integral on the unit sphere in $\mathbb{R}^n$ is zero, then the integral of kind (1.4) is called a singular integral operator of Calderon–Zygmund. (Note that this kind of operator was investigated earlier by F. Tricomi 1926 and 1938 and S.G. Mihlin 1962.) As in the case of the Hilbert transformation these operators determine the bounded map of the space to itself if, for $1 < p < \infty$ (i.e. if $f(x) \in L_p(\mathbb{R}^n)$), then there exists a constant $A_p$ such that Calderon–Zygmund's inequality

$$\| Kf \|_p \le A_p \| f \|_p$$

holds for $1 < p < \infty$.

We expound the proof of this inequality only for the operator (1.5) given by I. Vekua [IId]. The proof for the general case there is in Calderon and Zygmund (1952; 1956) and also in Schwarz (1971).

As previously, we restrict ourselves to consider the taste functions $C_0^\infty(\mathbb{C})$ which is dense in $L_p(\mathbb{C})$.

Let

$$T^*f = \frac{1}{2\pi} \iint_{\mathbb{C}} f(\zeta + z) \frac{d\xi\, d\eta}{\zeta|\zeta|} = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \iint_{\mathbb{C} - (|\zeta| < \varepsilon)} f(\zeta + z) \frac{d\xi\, d\eta}{\zeta|\zeta|}$$

where the integral again is understanding in the sense of Cauchy's principal value. Since the kernel of this integral passes the property of 'radial' symmetry we pass to polar coordinates $\zeta = \rho e^{i\theta}$. Then

$$T^*f = \frac{1}{2}\int_0^{\pi}\left(\frac{1}{\pi}\int_0^{\infty}\frac{f(z+\rho e^{i\theta})-f(z-\rho e^{i\theta})}{\rho}\,d\rho\right)e^{-i\theta}\,d\theta.$$

Hence

$$\|T^*f\|_p \le \frac{\pi}{2}\max\left\|\frac{1}{\pi}\int_0^{\infty}\frac{f(z+\rho e^{i\theta})-f(z-\rho e^{i\theta})}{\rho}\,d\rho\right\|_p$$

where

$$\|f\|_p = \Big(\iint_{\mathbb{C}}|f(z)|^p\,dx\,dy\Big)^{1/p}.$$

If we substitute $ze^{i\theta}$ for $z$, it is evident that the norm of the right-hand side is not changed and we obtain

$$\|T^*f\|_p \le \frac{\pi}{2}\max\ \|Hf_0\|_p,\quad f_0(z) = f(ze^{i\theta}).$$

But by M. Riesz's inequality (1.2),

$$\|Hf_0\|_p^p = \iint_{\mathbb{C}}|Hf_0(u+iv)|^p\,du\,dv \le A_p^p\int_{-\infty}^{\infty}dv\int_{-\infty}^{\infty}|f_0(u+iv)|^p\,du$$

$$\le A_p^p\|f_0\|_p^p.$$

Hence

$$\|T^*f\|_p \le \frac{\pi}{2}A_p\|f\|_p. \tag{1.6}$$

Since $f\in C_0^{\infty}(\mathbb{C})$ then integrating by parts, we obtain

$$T^*f(z) = -\frac{1}{\pi}\iint_{\mathbb{C}}f(\zeta+z)\frac{\partial}{\partial\zeta}\frac{1}{|\zeta|}\,d\xi\,d\eta$$