SQUARING THE CIRCLE

AND OTHER MONOGRAPHS

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BY E. W. HOBSON

RULER AND COMPASS

BY H. P. HUDSON

THE THEORY AND CONSTRUCTION OF NON-DIFFERENTIABLE FUNCTIONS

BY A. N. SINGH

HOW TO DRAW A STRAIGHT LINE; A LECTURE ON LINKAGES

BY A. B. KEMPE

CHELSEA PUBLISHING COMPANY

1953

SQUARING THE CIRCLE WAS ORIGINALLY PUBLISHED BY THE CAMBRIDGE UNIVERSITY PRESS

RULER AND COMPASS WAS ORIGINALLY PUBLISHED BY LONGMANS GREEN & COMPANY, INC.

THE THEORY AND CONSTRUCTION OF NON-DIFFERENTIABLE FUNCTIONS WAS ORIGINALLY PUBLISHED BY LUCKNOW UNIV. PRESS

HOW TO DRAW A STRAIGHT LINE WAS ORIGINALLY PUBLISHED BY MACMILLAN & COMPANY

PRINTED IN U.S.A.

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A HISTORY OF THE PROBLEM

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BY

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PREFACE

IN the Easter Term of the present year I delivered a short course of six Professorial Lectures on the history of the problem of the quadrature of the circle, in the hope that a short account of the fortunes of this celebrated problem might not only prove interesting in itself, but might also act as a stimulant of interest in the more general history of Mathematics. It has occurred to me that, by the publication of the Lectures, they might perhaps be of use, in the same way, to a larger circle of students of Mathematics.

The account of the problem here given is not the result of any independent historical research, but the facts have been taken from the writings of those authors who have investigated various parts of the history of the problem.

The works to which I am most indebted are the very interesting book by Prof. F. Rudio entitled "Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung" (Leipzig, 1892), and Sir T. L. Heath's treatise "The works of Archimedes" (Cambridge, 1897). I have also made use of Cantor's "Geschichte der Mathematik," of Vahlen's "Konstruktionen und Approximationen" (Leipzig, 1911), of Yoshio Mikami's treatise "The development of Mathematics in China and Japan" (Leipzig, 1913), of the translation by T. J. McCormack (Chicago, 1898) of H. Schubert's "Mathematical Essays and Recreations," and of the article "The history and transcendence of π" written by Prof. D. E. Smith which appeared in the "Monographs on Modern Mathematics" edited by Prof. J. W. A. Young. On special points I have consulted various other writings.

E. W. H.

CHRIST'S COLLEGE, CAMBRIDGE.
October, 1913.

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CHAPTER I

GENERAL ACCOUNT OF THE PROBLEM

A GENERAL survey of the history of thought reveals to us the fact of the existence of various questions that have occupied the almost continuous attention of the thinking part of mankind for long series of centuries. Certain fundamental questions presented themselves to the human mind at the dawn of the history of speculative thought, and have maintained their substantial identity throughout the centuries, although the precise terms in which such questions have been stated have varied from age to age in accordance with the ever varying attitude of mankind towards fundamentals. In general, it may be maintained that, to such questions, even after thousands of years of discussion, no answers have been given that have permanently satisfied the thinking world, or that have been generally accepted as final solutions of the matters concerned. It has been said that those problems that have the longest history are the insoluble ones.

If the contemplation of this kind of relative failure of the efforts of the human mind is calculated to produce a certain sense of depression, it may be a relief to turn to certain problems, albeit in a more restricted domain, that have occupied the minds of men for thousands of years, but which have at last, in the course of the nineteenth century, received solutions that we have reasons of overwhelming cogency to regard as final. Success, even in a comparatively limited field, is some compensation for failure in a wider field of endeavour. Our legitimate satisfaction at such exceptional success is but slightly qualified by the fact that the answers ultimately reached are in a certain sense of a negative character. We may rest contented with proofs that these problems, in their original somewhat narrow form, are insoluble, provided we attain, as is actually the case in some celebrated instances, to a complete comprehension of the grounds, resting upon a thoroughly established theoretical basis, upon which our final conviction of the insolubility of the problems is founded.

The three celebrated problems of the quadrature of the circle, the trisection of an angle, and the duplication of the cube, although all of them are somewhat special in character, have one great advantage for the purposes of historical study, *viz.* that their complete history as scientific problems lies, in a completed form, before us. Taking the first of these problems, which will be here our special subject of study, we possess indications of its origin in remote antiquity, we are able to follow the lines on which the treatment of the problem proceeded and changed from age to age in accordance with the progressive development of general Mathematical Science, on which it exercised a noticeable reaction. We are also able to see how the progress of endeavours towards a solution was affected by the intervention of some of the greatest Mathematical thinkers that the world has seen, such men as Archimedes, Huyghens, Euler, and Hermite. Lastly, we know when and how the resources of modern Mathematical Science became sufficiently powerful to make possible that resolution of the problem which, although negative, in that the impossibility of the problem subject to the implied restrictions was proved, is far from being a mere negation, in that the true grounds of the impossibility have been set forth with a finality and completeness which is somewhat rare in the history of Science.

If the question be raised, why such an apparently special problem, as that of the quadrature of the circle, is deserving of the sustained interest which has attached to it, and which it still possesses, the answer is only to be found in a scrutiny of the history of the problem, and especially in the closeness of the connection of that history with the general history of Mathematical Science. It would be difficult to select another special problem, an account of the history of which would afford so good an opportunity of obtaining a glimpse of so many of the main phases of the development of general Mathematics; and it is for that reason, even more than on account of the intrinsic interest of the problem, that I have selected it as appropriate for treatment in a short course of lectures.

Apart from, and alongside of, the scientific history of the problem, it has a history of another kind, due to the fact that, at all times, and almost as much at the present time as formerly, it has attracted the attention of a class of persons who have, usually with a very inadequate equipment of knowledge of the true nature of the problem or of its history, devoted their attention to it, often with passionate enthusiasm. Such persons have very frequently maintained, in the face of all efforts at refutation made by genuine Mathematicians, that they had obtained a solution of the problem. The solutions propounded by the circle squarer exhibit every grade of skill, varying from the most futile attempts, in which the writers shew an utter lack of power to reason correctly, up to approximate solutions the construction of which required much ingenuity on the part of their inventor. In some cases it requires an effort of sustained attention to find out the precise point in the demonstration at which the error occurs, or in which an approximate determination is made to do duty for a theoretically exact one. The psychology of the scientific crank is a subject with which the officials of every Scientific Society have some practical acquaintance. Every Scientific Society still receives from time to time communications from the circle squarer and the trisector of angles, who often make amusing attempts to disguise the real character of their essays. The solutions propounded by such persons usually involve some misunderstanding as to the nature of the conditions under which the problems are to be solved, and ignore the difference between an approximate construction and the solution of the ideal problem. It is a common occurrence that such a person sends his solution to the authorities of a foreign University or Scientific Society, accompanied by a statement that the men of Science of the writer's own country have entered into a conspiracy to suppress his work, owing to jealousy, and that he hopes to receive fairer treatment abroad. The statement is not infrequently accompanied with directions as to the forwarding of any prize of which the writer may be found worthy by the University or Scientific Society addressed, and usually indicates no lack of confidence that the bestowal of such a prize has been amply deserved as the fit reward for the final solution of a problem which has baffled the efforts of a great multitude of predecessors in all ages. A very interesting detailed account of the peculiarities of the circle squarer, and of the futility of attempts on the part of Mathematicians to convince him of his errors, will be found in Augustus De Morgan's *Budget of Paradoxes*. As early as the time of the Greek Mathematicians circle-squaring occupied the attention of non-Mathematicians; in fact the Greeks had a special word to denote this kind of activity, *viz.* τετραγωνίζειν, which means to occupy oneself with the quadrature. It is interesting to remark that, in the year 1775, the Paris Academy found it necessary to protect its officials against the waste of time and energy involved in examining the efforts of circle squarers. It passed a resolution, which appears in the Minutes of the Academy*, that no more solutions were to be examined of the problems of the duplication of the cube, the trisection of the angle, the quadrature of the circle, and that the same resolution should apply to machines for exhibiting perpetual motion. An account of the reasons which led to the adoption of this resolution, drawn up by Condorcet, who was then the perpetual Secretary of the Academy, is appended. It is interesting to remark the strength of the conviction of Mathematicians that the solution of the problem is impossible, more than a century before an irrefutable proof of the correctness of that conviction was discovered.

The popularity of the problem among non-Mathematicians may seem to require some explanation. No doubt, the fact of its comparative obviousness explains in part at least its popularity; unlike many Mathematical problems, its nature can in some sense be understood by anyone; although, as we shall presently see, the very terms in which it is usually stated tend to suggest an imperfect apprehension of its precise import. The accumulated celebrity which the problem attained, as one of proverbial difficulty, makes it an irresistible attraction to men with a certain kind of mentality. An exaggerated notion of the gain which would accrue to mankind by a solution of the problem has at various times been a factor in stimulating the efforts of men with more zeal than knowledge. The man of mystical tendencies has been attracted to the problem by a vague idea that its solution would, in some dimly discerned manner, prove a key to a knowledge of the inner connections of things far beyond those with which the problem is immediately connected.

Statement of the problem

The fact was well known to the Greek Geometers that the problems of the quadrature and the rectification of the circle are equivalent problems. It was in fact at an early time established that the ratio of the length of a complete circle to the diameter has a definite value equal to that of the area of the circle to that of a square of which the radius is side. Since the time of Euler this ratio has always been denoted by the familiar notation π. The problem of "squaring the circle" is roughly that of constructing a square of which the area is equal to that enclosed by the circle. This is then equivalent to the problem of the rectification of the circle, *i.e.* of the determination of a

* *Histoire de l'Académie royale*, année 1775, p. 61.

straight line, of which the length is equal to that of the circumference of the circle. But a problem of this kind becomes definite only when it is specified what means are to be at our disposal for the purpose of making the required construction or determination; accordingly, in order to present the statement of our problem in a precise form, it is necessary to give some preliminary explanations as to the nature of the postulations which underlie all geometrical procedure.

The Science of Geometry has two sides; on the one side, that of practical or physical Geometry, it is a physical Science concerned with the actual spatial relations of the extended bodies which we perceive in the physical world. It was in connection with our interests, of a practical character, in the physical world, that Geometry took its origin. Herodotus ascribes its origin in Egypt to the necessity of measuring the areas of estates of which the boundaries had been obliterated by the inundations of the Nile, the inhabitants being compelled, in order to settle disputes, to compare the areas of fields of different shapes. On this side of Geometry, the objects spoken of, such as points, lines, &c., are physical objects; a point is a very small object of scarcely perceptible and practically negligible dimensions; a line is an object of small, and for some purposes negligible, thickness; and so on. The constructions of figures consisting of points, straight lines, circles, &c., which we draw, are constructions of actual physical objects. In this domain, the possibility of making a particular construction is dependent upon the instruments which we have at our disposal.

On the other side of the subject, Geometry is an abstract or rational Science which deals with the relations of objects that are no longer physical objects, although these ideal objects, points, straight lines, circles, &c., are called by the same names by which we denote their physical counterparts. At the base of this rational Science there lies a set of definitions and postulations which specify the nature of the relations between the ideal objects with which the Science deals. These postulations and definitions were suggested by our actual spatial perceptions, but they contain an element of absolute exactness which is wanting in the rough data provided by our senses. The objects of abstract Geometry possess in absolute precision properties which are only approximately realized in the corresponding objects of physical Geometry. In every department of Science there exists in a greater or less degree this distinction between the abstract or rational side and the physical or concrete side; and the progress of each department of Science involves a continually increasing amount of rationalization. In Geometry the passage from a purely empirical treatment to the setting up of a rational Science proceeded by much more rapid stages than in other cases. We have in the Greek Geometry, known to us all through the presentation of it given in that oldest of all scientific text books, Euclid's *Elements of Geometry*, a treatment of the subject in which the process of rationalization has already reached an advanced stage. The possibility of solving a particular problem of determination, such as the one we are contemplating, as a problem of rational Geometry, depends upon the postulations that are made as to the allowable modes of determination of new geometrical elements by means of assigned ones. The restriction in practical Geometry to the use of specified instruments has its counterpart in theoretical Geometry in restrictions as to the mode in which new elements are to be determined by means of given ones. As regards the postulations of rational Geometry in this respect there is a certain arbitrariness corresponding to the more or less arbitrary restriction in practical Geometry to the use of specified instruments.

The ordinary obliteration of the distinction between abstract and physical Geometry is furthered by the fact that we all of us, habitually and almost necessarily, consider both aspects of the subject at the same time. We may be thinking out a chain of reasoning in abstract Geometry, but if we draw a figure, as we usually must do in order to fix our ideas and prevent our attention from wandering owing to the difficulty of keeping a long chain of syllogisms in our minds, it is excusable if we are apt to forget that we are not in reality reasoning about the objects in the figure, but about objects which are their idealizations, and of which the objects in the figure are only an imperfect representation. Even if we only visualize, we see the images of more or less gross physical objects, in which various qualities irrelevant for our specific purpose are not entirely absent, and which are at best only approximate images of those objects about which we are reasoning.

It is usually stated that the problem of squaring the circle, or the equivalent one of rectifying it, is that of constructing a square of an area equal to that of the circle, or in the latter case of constructing a straight line of length equal to that of the circumference, by a method which involves the use only of the compass and of the ruler as a single straight-edge. This mode of statement, although it indicates roughly the true statement of the problem, is decidedly defective in that it entirely leaves out of account the fundamental distinction between the two aspects of Geometry to which allusion has been made above. The compass and the straight-edge are physical objects by the use of which other objects can be constructed, *viz.* circles of small thickness, and lines which are approximately straight and very thin, made of ink or other material. Such instruments can clearly have no direct relation to theoretical Geometry, in which circles and straight lines are ideal objects possessing in absolute precision properties that are only approximately realized in the circles and straight lines that can be constructed by compasses and rulers. In theoretical Geometry, a restriction to the use of rulers and compasses, or of other instruments, must be replaced by corresponding postulations as to the allowable modes of determination of geometrical objects. We will see what these postulations really are in the case of Euclidean Geometry. Every Euclidean problem of construction, or as it would be preferable to say, every problem of determination, really consists in the determination of one or more points which shall satisfy prescribed conditions. We have here to consider the fundamental modes in which, when a number of points are regarded as given, or already determined, a new point is allowed to be determined.

Two of the fundamental postulations of Euclidean Geometry are that, having given two points A and B, then (1) a unique straight line (A, B) (the whole straight line, and not merely the segment between A and B) is determined such that A and B are incident on it, and (2) that a unique circle $A\,(B)$, of which A is centre and on which B is incident, is determined. The determinancy or assumption of existence of such straight lines and circles is in theoretical Geometry sufficient for the purposes of the subject. When we know that these objects, having known properties, exist, we may reason about them and employ them for the purposes of our further procedure; and that is sufficient for our purpose. The notion of drawing or constructing them by means of a straight-edge or compass has no relevance to abstract Geometry, but is borrowed from the language of practical Geometry.

A new point is determined in Euclidean Geometry exclusively in one of the three following ways:

Having given four points A, B, C, D, not all incident on the same straight line, then

(1) Whenever a point P exists which is incident both on (A, B) and on (C, D), that point is regarded as determinate.

(2) Whenever a point P exists which is incident both on the straight line (A, B) and on the circle $C(D)$, that point is regarded as determinate.

(3) Whenever a point P exists which is incident on both the circles $A(B)$, $C(D)$, that point is regarded as determinate.

The cardinal points of any figure determined by a Euclidean construction are always found by means of a finite number of successive applications of some or all of these rules (1), (2) and (3). Whenever one of these rules is applied it must be shewn that it does not fail to determine the point. Euclid's own treatment is sometimes defective as regards this requisite; as for example in the first proposition of his first book, in which it is not shewn that the circles intersect one another.

In order to make the practical constructions which correspond to these three Euclidean modes of determination, corresponding to (1) the ruler is required, corresponding to (2) both the ruler and the compass, and corresponding to (3) the compass only.

As Euclidean plane Geometry is concerned with the relations of points, straight lines, and circles only, it is clear that the above system of postulations, although arbitrary in appearance, is the system that the exigencies of the subject would naturally suggest. It may, however, be remarked that it is possible to develop Euclidean Geometry with a more restricted set of postulations. For example it can be shewn that all Euclidean constructions can be carried out by means of (3) alone*, without employing (1) or (2).

Having made these preliminary explanations we are now in a position to state in a precise form the ideal problem of "squaring the circle," or the equivalent one of the rectification of the circle.

The historical problem of "squaring the circle" is that of determining a square of which the area shall equal that of a given circle, by a method such that the determination of the corners of the square is to be made by means of the above rules (1), (2), (3), each of which may be applied any finite number of times. In other words, each new point successively determined in the process of construction is to be obtained as the intersection of two straight lines already determined, or as an intersection of a straight line and a circle already determined, or as an intersection of two circles already determined. A

* See for example the *Mathematical Gazette* for March 1913, where I have treated this point in detail in the Presidential Address to the Mathematical Association.

similar statement applies to the equivalent problem of the rectification of the circle.

This mode of determination of the required figure we may speak of shortly as a Euclidean determination.

Corresponding to any problem of Euclidean determination there is a practical problem of physical Geometry to be carried out by actual construction of straight lines and circles by the use of ruler and compasses. Whenever an ideal problem is soluble as one of Euclidean determination the corresponding practical problem is also a feasible one. The ideal problem has then a solution which is ideally perfect; the practical problem has a solution which is an approximation limited only by the imperfections of the instruments used, the ruler and the compass; and this approximation may be so great that there is no perceptible defect in the result. But it is an error which accounts I think, in large measure, for the aberrations of the circle squarer and the trisector of angles, to assume the converse that, when a practical problem is soluble by the use of the instruments in such a way that the error is negligible or imperceptible, the corresponding ideal problem is also soluble. This is very far from being necessarily the case. It may happen that in the case of a particular ideal problem no solution is obtainable by a finite number of successive Euclidean determinations, and yet that such a finite set gives an approximation to the solution which may be made as close as we please by taking the process far enough. In this case, although the ideal problem is insoluble by the means which are permitted, the practical problem is soluble in the sense that a solution may be obtained in which the error is negligible or imperceptible, whatever standard of possible perceptions we may employ. As we have seen, a Euclidean problem of construction is reducible to the determination of one or more points which satisfy prescribed conditions. Let P be one such point; then it may be possible to determine in Euclidean fashion each point of a set $P_1, P_2, \dots P_n, \dots$ of points which converge to P as limiting point, and yet the point P may be incapable of determination by Euclidean procedure. This is what we now know to be the state of things in the case of our special problem of the quadrature of the circle by Euclidean determination. As an ideal problem it is not capable of solution, but the corresponding practical problem is capable of solution with an accuracy bounded only by the limitations of our perceptions and the imperfections of the instruments employed. Ideally we can actually determine by Euclidean methods a square of which the area differs from that of a given circle by less than an arbitrarily prescribed magnitude, although we cannot pass to the limit. We can obtain solutions of the corresponding physical problem which leave nothing to be desired from the practical point of view. Such is the answer which has been obtained to the question raised in this celebrated historical problem of Geometry. I propose to consider in some detail the various modes in which the problem has been attacked by people of various races, and through many centuries; how the modes of attack have been modified by the progressive development of Mathematical tools, and how the final answer, the nature of which had been long anticipated by all competent Mathematicians, was at last found and placed on a firm basis.

General survey of the history of the problem

The history of our problem is typical as exhibiting in a remarkable degree many of the phenomena that are characteristic of the history of Mathematical Science in general. We notice the early attempts at an empirical solution of the problem conceived in a vague and sometimes confused manner; the gradual transition to a clearer notion of the problem as one to be solved subject to precise conditions. We observe also the intimate relation which the mode of regarding the problem in any age had with the state then reached by Mathematical Science in its wider aspect; the essential dependence of the mode of treatment of the problem on the powers of the existing tools. We observe the fact that, as in Mathematics in general, the really great advances, embodying new ideas of far-reaching fruitfulness, have been due to an exceedingly small number of great men; and how a great advance has often been followed by a period in which only comparatively small improvements in, and detailed developments of, the new ideas have been accomplished by a series of men of lesser rank. We observe that there have been periods when for a long series of centuries no advance was made; when the results obtained in a more enlightened age have been forgotten. We observe the times of revival, when the older learning has been rediscovered, and when the results of the progress made in distant countries have been made available as the starting points of new efforts and of a fresh period of activity.

The history of our problem falls into three periods marked out by fundamentally distinct differences in respect of method, of immediate aims, and of equipment in the possession of intellectual tools. The first period embraces the time between the first records of empirical