PLANE ALGEBRAIC CURVES

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PREFACE

Plane Algebraic Curves is based on lectures which I have given to honours mathematics students at the University College of Leicester during the last few years. Its aim is to give the student who is unfamiliar with the theory of curves a reasonably brief introduction to the subject: he should then be in a position to study the more advanced works, some of which are recommended at the end of the book.

It may fairly be said that there is a need for such a book. There is no elementary book on curves which is still in print, and works such as Coolidge's *Algebraic Plane Curves* and the recent book by Walker on *Algebraic Curves* are too difficult for a beginner.

I assume that the reader has studied analytical geometry, using Cartesian coordinates, as far as the theory of conics. I assume also that he knows a little about homogeneous coordinates, including the principle of duality: Chapter I of Maxwell's book on *General Homogeneous Coordinates* will be sufficient for this purpose, except in some of the later examples.

Though the book is intended for honours mathematics students at a university, the first few chapters could be studied with profit by advanced sixth-form pupils in schools.

I should like to thank various bodies for permission to use examination questions. Questions taken from Oxford University examination papers are reproduced by permission of the Clarendon Press, Oxford, and are denoted by O. Those from Cambridge Tripos papers are denoted by M.T., those from University of London papers by L., and those from the Oxford and Cambridge Joint Board Certificate papers by O.C.

There are examples for solution at the end of Chapters I, II, III, IV, VI and VII, and a set of general examples at the end of the book. Full solutions of all these are given finally. Difficult examples are marked with an asterisk.

I am indebted to the publishers for their courtesy and patience in dealing with my requests, and to the printers for the excellence of their work.

E. J. F. PRIMROSE

CONTENTS

CHAPTER I

CURVES IN THE REAL EUCLIDEAN PLANE

1.1. The days when great time and trouble were taken in tracing particular curves, often complicated, are past. To-day the main emphasis is on powerful methods and results which apply to curves in general. However, a small amount of curve-tracing is useful in order to make the student familiar with some of the properties of curves.

By curve-tracing we mean giving a rough diagram of a curve, showing its most important features. We shall confine ourselves to Euclidean geometry and to the field of real numbers, except when we explicitly mention complex numbers. Before stating the most important features of curves, we must give some definitions.

1.2. (i) The *degree* n of a curve is the highest power of x and y combined which occurs in its equation: in homogeneous coordinates n is the degree of every term of the equation. For example, the curve $x^2y=1$ is of degree 3, because the term x^2y is of degree 3 in x and y combined: in homogeneous coordinates, the equation is $x^2y=z^3$, in which both the terms are of degree 3.

In the field of complex numbers the degree n is the number of points in which the curve is met by any line provided that (i) points at infinity are included and (ii) coincident points are counted the appropriate number of times. By this we mean that if the equation of the curve and the equation of the line are solved (by eliminating one of the variables) and if an r-fold root occurs in the resulting equation then the corresponding point is counted r times. There are two ways in which a multiple root may arise: either the line may touch the curve or it may pass through a multiple point of the curve (see (ii) below).

In the field of real numbers the number of intersections may be less than n because the equation obtained by eliminating one of the variables may have less than n roots. However, the number must be of the form $n-2h$, where h is a non-negative integer, because complex roots of a real equation occur in pairs.

As an example, we consider the intersections of the curve

$$y^2(y-1)^2=(x-y)(x^2-2xy+3y^2)$$

with the line $x-y=0$. Eliminating x, we obtain $y^2(y-1)^2=0$: hence the line meets the curve twice at (0, 0) and twice at (1, 1). After discussing multiple points we could show that (0, 0) is a double point and (1, 1) is a simple point at which $x-y=0$ is the tangent.

(ii) A *double point* of a curve is a point P such that *every* line through P meets the curve twice at P. Similarly a *multiple point of order* r, or r-fold point ($r>1$) is a point P such that every line through P meets the curve r times at P. A point which is not multiple is called *simple*. (We shall see later that certain lines meet the curve more than r times at a multiple point P.)

(iii) An *asymptote* is a finite line which touches the curve at infinity. An example is either of the coordinate axes for the hyperbola $xy=c^2$.

1.3. We now give a list of the most important features of curves. The first group, (i) to (iii), are properties of the curves themselves and the second group, (iv) to (vi), depend on the choice of coordinate axes.

They are as follows:

(i) The multiple points.

(ii) The way in which the curve tends to infinity. This means finding the asymptotes and on which side of them the curve tends to infinity, and branches tending to infinity but not approaching an asymptote (as the parabola does, for example): such branches will be called *non-linear*.

(iii) Symmetry of the curve.

(iv) The points where the curve cuts the axes.

(v) The form of the curve near the origin.

(vi) Ranges of values of x which give no real value for y, and vice versa.

We now examine some of these features in detail.

1.4. Multiple points

We first investigate the different types of double point. Let the equation of the curve be written in ascending powers of x and y in the form

$$a+(bx+cy)+\frac{1}{2}(dx^2+2exy+fy^2)+\ldots=0,$$

and let the origin O be a double point. Any line $y=mx$ through O cuts the curve in points given by

$$a+(b+cm)x+\frac{1}{2}(d+2em+fm^2)x^2+\ldots=0.$$

This equation must have two roots $x=0$ for all values of m, and so $a=0$, $b=0$, $c=0$. If, in addition, $d+2em+fm^2=0$, but d, e, f are not all zero, the line $y=mx$ (but not every line through O) meets the curve three times at O. Eliminating m between the two equations, we obtain

$$dx^2+2exy+fy^2=0, \tag{1}$$

that is, the equation formed by equating the quadratic terms to zero. Three cases arise:

(i) $df-e^2<0$, when the equation represents two lines through O,

(ii) $df-e^2>0$, when the equation represents the point O,

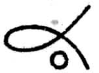
(iii) $df-e^2=0$, when the equation represents one line (or two coincident lines) through O.

Of course, in the field of complex numbers, (i) and (ii) both give two lines through O.

We now show that the lines (if any) given by (1) are the tangents at O. Let Q be a point of the curve other than O. Then OQ meets the curve twice at O and once at Q (and possibly at other points). If Q tends to O, then in the limit OQ meets the curve three times at O and is, by the usual definition of tangent, a tangent at O.

The three cases may be drawn thus:

(i)



The curve crosses itself at O and has two distinct tangents there. O is called a *crunode*. An example is the origin on the curve $(x^2+y^2)^2=a^2(x^2-y^2)$.

(ii)

The curve has an isolated point at O, since there are no tangents there. O is called an *acnode*. An example is the origin on the curve $(x^2-y^2)^2=a^2(x^2+y^2)$.

(iii)

There is one tangent at O, and O is called a *cusp*. An example is the origin on the curve $y^2=x^3$. There are also more complicated types of cusp, which will be considered in detail in Chapter IV.

(i) and (ii) are both called *nodes*.

We now show how to find all the double points of a curve, including those at infinity. In order to do this we use homogeneous Cartesian coordinates. We first need a theorem which will be useful again later.

The function $\partial/\partial x\{f(x, y, z)\}$, evaluated for the values $x=X$, $y=Y$, $z=Z$, will be denoted by $\partial f/\partial X$, or f_X.

THEOREM 1

If $P(X, Y, Z)$ is a point on the curve $f(x, y, z)=0$, where $f(x, y, z)$ is a homogeneous polynomial in x, y, z of degree n, and $Q(x, y, z)$ is another point of the plane, then if any point on PQ is expressed in the form $(\lambda X+\mu x, \lambda Y+\mu y, \lambda Z+\mu z)$, the intersections of PQ with the curve are given by

$$\lambda^{n-1}\mu(xf_X+yf_Y+zf_Z)+\tfrac{1}{2}\lambda^{n-2}\mu^2(x^2f_{XX}+y^2f_{YY}+z^2f_{ZZ}+2yzf_{YZ} + 2zxf_{ZX}+2xyf_{XY})+\ldots=0. \quad (2)$$

Proof. The intersections with the curve are given by

$$f(\lambda X+\mu x, \lambda Y+\mu y, \lambda Z+\mu z)=0.$$

Using Taylor's theorem for a function of three variables, we have

$$f(\lambda X, \lambda Y, \lambda Z)+\mu\Sigma xf_x(\lambda X, \lambda Y, \lambda Z)+\tfrac{1}{2}\mu^2[\Sigma x^2f_{xx}(\lambda X, \lambda Y, \lambda Z) + 2\Sigma yzf_{yz}(\lambda X, \lambda Y, \lambda Z)]+\ldots=0,$$

where Σ stands for a summation over x, y, z. (Since f is a polynomial in x, y, z, the expansion terminates.) Since f is a homogeneous function of x, y, z, of degree n, we have

$$f(\lambda X, \lambda Y, \lambda Z) = \lambda^n f(X, Y, Z) = 0, \text{ since } P \text{ lies on the curve,}$$
$$f_x(\lambda X, \lambda Y, \lambda Z) = \lambda^{n-1} f_x(X, Y, Z) = \lambda^{n-1} f_X,$$

and so on. The result now follows immediately.

Corollary. The tangent to the curve at a simple point (X, Y, Z) is

$$x f_X + y f_Y + z f_Z = 0.$$

For, if Q satisfies this equation, PQ meets the curve twice at P, and since P is a simple point PQ must be the tangent at P.

Now if (X, Y, Z) is a double point, equation (2) must have two roots $\mu = 0$ for all x, y, z, since PQ meets the curve twice at P. Hence

$$f_X = 0, f_Y = 0, f_Z = 0.$$

(If the coordinates of a point satisfy these equations they automatically satisfy the equation of the curve since, by Euler's theorem on homogeneous functions, $nf = xf_x + yf_y + zf_z$.) The reader should notice that we have three equations to solve for the two independent ratios of X, Y, Z, so that a general curve has no double points.

If $\Sigma x^2 f_{XX} + 2\Sigma yz f_{YZ} = 0$, equation (2) has three roots $\mu = 0$, so the line PQ meets the curve three times at P. This equation therefore gives the tangents (if any) at P.

EXAMPLE

Find the positions and types of the double points of the curve

$$f(x, y, z) = x^2 y^2 - 2z^3(x + y) + 3z^4 = 0.$$

For a double point,

$$f_x \equiv 2(xy^2 - z^3) = 0, \qquad (3)$$
$$f_y \equiv 2(x^2 y - z^3) = 0, \qquad (4)$$
$$f_z \equiv 6z^2(-x - y + 2z) = 0. \qquad (5)$$

Hence either

(i) $z = 0$, in which case either $x = 0$ or $y = 0$, or
(ii) $x = y$, from (3) and (4), and $x = y = z$, from (5).

Hence there are two double points at infinity,* (1, 0, 0) and (0, 1, 0), and a finite double point, (1, 1, 1). Now

$$f_{xx}=2y^2,\ f_{yy}=2x^2,\ f_{zz}=12z(-x-y+3z),$$
$$f_{yz}=-6z^2,\ f_{zx}=-6z^2,\ f_{xy}=4xy.$$

At (1, 0, 0), $f_{yy}=2$, and all the others are zero. Hence the tangents at (1, 0, 0) are given by $y^2=0$. These are coincident, so (1, 0, 0) is a cusp with tangent $y=0$. Similarly (0, 1, 0) is a cusp with tangent $x=0$. At (1, 1, 1), $f_{xx}=2, f_{yy}=2, f_{zz}=12$, $f_{yz}=-6$, $f_{zx}=-6$, $f_{xy}=4$. The tangents at (1, 1, 1) are therefore given by

$$x^2+y^2+6z^2-6yz-6zx+4xy=0,$$

i.e. $$3(x+y)\pm\sqrt{3}(x-y)-6z=0.$$

Hence the point (1, 1, 1) is a crunode.

In finding double points it is helpful to assume the result of a theorem which we shall prove later, that a non-degenerate curve of degree n (that is, a curve which does not split up into two or more curves) cannot have more than $\frac{1}{2}(n-1)(n-2)$ double points.

If, as often happens, the curve is given in the non-homogeneous form, and only the finite double points are required (the double points at infinity can be found by another method, as in the following section) it is easier to proceed as follows.

Since $$nf\equiv xf_x+yf_y+zf_z,$$

any point whose coordinates satisfy the equations $f=0$, $f_x=0$, $f_y=0$ also satisfies the equation $zf_z=0$. If the point is finite, $z\neq 0$, so $f_z=0$. Hence the finite double points are given by $f=0$, $f_x=0$, $f_y=0$, which can all be written in the non-homogeneous form.

In the example above, finite double points are given by

$$f(x, y)\equiv x^2y^2-2(x+y)+3=0$$
$$f_x\equiv 2xy^2-2=0$$
$$f_y\equiv 2x^2y-2=0.$$

* The interpretation of a double point at infinity will be given later (1.5 exs. (ii), (iv)).

From the last two equations $x=y=1$, which satisfies the first equation. Hence (1, 1) is a double point.

In order to find the tangents at the double point, we transfer the origin to the point (1, 1) by putting $x=X+1$, $y=Y+1$. The equation of the curve then becomes

$$(X+1)^2(Y+1)^2-2(X+Y+2)+3=0.$$

The tangents are given by equating the terms of lowest degree to zero, which gives

$$X^2+4XY+Y^2=0,$$

or, in terms of the original coordinates,

$$(x-1)^2+4(x-1)(y-1)+(y-1)^2=0.$$

Multiple points of higher order may be studied in the same way. The reader will be able to prove the following results for a multiple point at the origin :

(i) the degree of the terms of lowest degree in x and y combined is the order of the multiple point,

(ii) the tangents at the multiple point are given by equating the terms of lowest degree in x and y to zero.

1.5. The asymptotes of a curve

THEOREM 2

Let the equation of a curve be written, in non-homogeneous Cartesian coordinates, in the form

$$f(x, y)\equiv f_n(x, y)+f_{n-1}(x, y)+\ldots+f_0=0,$$

where $f_r(x, y)$ is a homogeneous polynomial in x and y of degree r. Then

(i) if $f_n(x, y)$ contains a simple factor $ax+by$, so that $f_n(x, y)\equiv (ax+by)\phi_{n-1}(x, y)$, the curve has an asymptote

$$(ax+by)\phi_{n-1}(b, -a)+f_{n-1}(b, -a)=0, \qquad (1)$$

(ii) if $f_n(x, y)$ contains a repeated factor $ax+by$, and $f_{n-1}(x, y)$ also contains a factor $ax+by$, so that $f_n(x, y)\equiv(ax+by)^2\phi_{n-2}(x, y)$, $f_{n-1}(x, y)\equiv(ax+by)\psi_{n-2}(x, y)$, the curve has a pair of parallel asymptotes whose combined equation is

$$(ax+by)^2\phi_{n-2}(b, -a)+(ax+by)\psi_{n-2}(b, -a)+f_{n-2}(b, -a)=0, \qquad (2)$$

(iii) *if $f_n(x, y)$ contains a repeated factor $ax+by$, and $f_{n-1}(x, y)$ does not contain a factor $ax+by$, there is no asymptote in the direction $ax+by=0$.*

Proof. We make the equation homogeneous in x, y, z, so that it is

$$f(x, y, z) \equiv f_n(x, y) + zf_{n-1}(x, y) + \dots + z^n f_0 = 0.$$

In each case there is a point at infinity $(b, -a, 0)$.

In (i), at $(b, -a, 0)$, $\partial f/\partial x = a\phi_{n-1}(b, -a)$, $\partial f/\partial y = b\phi_{n-1}(b, -a)$, $\partial f/\partial z = f_{n-1}(b, -a)$. The equation of the tangent at $(b, -a, 0)$ is therefore (1).

In (ii), at $(b, -a, 0)$, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, $\partial f/\partial z = 0$, so there is a double point at $(b, -a, 0)$. Now

$$\partial^2 f/\partial x^2 = 2a^2\phi_{n-2}(b, -a), \quad \partial^2 f/\partial y^2 = 2b^2\phi_{n-2}(b, -a),$$
$$\partial^2 f/\partial z^2 = 2f_{n-2}(b, -a), \quad \partial^2 f/\partial y\partial z = b\psi_{n-2}(b, -a),$$
$$\partial^2 f/\partial z\partial x = a\psi_{n-2}(b, -a), \quad \partial^2 f/\partial x\partial y = 2ab\phi_{n-2}(b, -a).$$

The equation of the tangents at $(b, -a, 0)$ is therefore (2). In the special case where

$$\psi^2_{n-2}(b, -a) = 4\phi_{n-2}(b, -a) f_{n-2}(b, -a),$$

the tangents coincide, so the double point is a cusp.

In (iii), at $(b, -a, 0)$, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, $\partial f/\partial z = f_{n-1}(b, -a) \neq 0$, so the tangent at $(b, -a, 0)$ is $z=0$.

EXAMPLES (i) $xy(x+y) + xy + y^2 + 3x = 0$. (L.)

At the point at infinity $(0, 1, 0)$, corresponding to the factor x in the terms of the highest degree, we have $a=1$, $b=0$, $\phi_2 \equiv y(x+y)$, $f_2 \equiv y(x+y)$. Using (1), we see that the asymptote is $x+1=0$. For the point $(1, 0, 0)$, $a=0$, $b=1$, $\phi_2 \equiv x(x+y)$, $f_2 \equiv y(x+y)$. The asymptote is $y=0$. For the point $(1, -1, 0)$, $a=1$, $b=1$, $\phi_2 \equiv xy$, $f_2 \equiv y(x+y)$. The asymptote is $x+y=0$.

(ii) $x(x+y)^2 - x(x+y) + 1 = 0$.

The point $(0, 1, 0)$ is treated as in (i). For the point $(1, -1, 0)$, $a=1$, $b=1$, $\phi_1 \equiv x$, $\psi_1 \equiv -x$, $f_1 \equiv 0$. Using (2), we see that the asymptotes are

$$(x+y)^2 - (x+y) = 0,$$

i.e. $$x+y=0 \text{ or } 1.$$

$$x^2y^2 + xy + x + 3y + 3 = 0. \quad \text{(L.)}$$

For the point (0, 1, 0), $a=1$, $b=0$, $\phi_2 \equiv y^2$, $\psi_2 \equiv 0$, $f_2 \equiv xy$. The asymptotes are therefore $x^2=0$, so there is a cusp at (0, 1, 0). The point (1, 0, 0) is treated similarly.

(iii) $(x+y)(x-y)^2 = 3xy - y^2$.

The point (1, −1, 0) is treated as in (i). Since the quadratic terms do not contain a factor $x-y$, there is no asymptote in the direction $x-y=0$.

Although Theorem 2 gives the asymptotes quite easily, it does not tell us on which side of the asymptote the curve lies at each end. We now give two methods which overcome this difficulty.

(1) One method, which is straightforward though rather long, is to assume that, for sufficiently large values of x, the equation of the curve can be expressed in the form

$$y = mx + c + \alpha/x + \beta/x^2 + \dots,$$

where the right-hand side is a convergent series. If this is so, then if we substitute this value for y in the equation of the curve, the resulting equation in x must be satisfied identically. We therefore equate the coefficients of the different powers of x to zero, thus obtaining equations for the coefficients m, c, α, β, etc.

Now, as x tends to infinity, the value of $y-mx-c$ tends to zero, so the asymptote is $y=mx+c$. The difference between y on the curve and y on the asymptote (for a given x) is given by the terms $\alpha/x + \beta/x^2 + \dots$, of which the first non-vanishing term is the most significant. We shall see in examples how this enables us to find on which side of the asymptote the curve lies.

This process breaks down in three cases. Firstly, if an asymptote is parallel to the y axis, m would have to be infinite. In this case we write $x = c + \alpha/y + \beta/y^2 + \dots$ and proceed as before. Secondly, if a branch tends to infinity but does not approach an asymptote, the equation of the curve cannot be expressed in the given form, and a contradiction arises. Thirdly, if a curve has branches going to the same end of an asymptote, again the equation cannot be expressed in the given form. We shall see examples of these later.

EXAMPLES (i) $xy(x+y)+xy+y^2+3x=0$.

The directions of the asymptotes are given by $x=0$, $y=0$, $x+y=0$. For the last two, we put

$$y=mx+c+\alpha/x+\beta/x^2+\ldots.$$

Equating to zero the coefficients of powers of x, we have,

for x^3: $m(m+1)=0$. This gives the directions again.

for x^2: $c(2m+1)+m+m^2=0$. This gives $c=0$ whether $m=0$ or -1.

for x: $(2m+1)\alpha+c^2+c+2mc+3=0$. If $m=0$, $\alpha=-3$: if $m=-1$, $\alpha=3$.

Hence $$y=-3/x+\ldots \text{ or } y=-x+3/x+\ldots$$

The asymptotes are $y=0$ and $x+y=0$. We now find on which side of the asymptotes the curve lies.

(*a*) $y=0$. When x is large and positive, y for the curve is less than y for the asymptote. Hence the curve lies below the asymptote at that end. When x is large and negative, y for the curve is greater than y for the asymptote. Hence the curve lies above the asymptote at that end.

(*b*) $x+y=0$. When x is large and positive, y for the curve is greater than y for the asymptote. Hence the curve lies above the asymptote at that end. Similarly, the curve lies below the asymptote at the other end.

We now find the asymptote parallel to $x=0$. We put

$$x=c+\alpha/y+\beta/y^2+\ldots$$

Equating to zero the coefficients of powers of y, we have

for y^2: $c+1=0$.

for y: $c^2+\alpha+c=0$. So $\alpha=0$.

for y^0: $2c\alpha+\beta+\alpha+3c=0$. So $\beta=3$.

Hence $$x=-1+3/y^2+\ldots$$

The asymptote is $x=-1$. When y is large, whether positive or negative, x for the curve is greater than x for the asymptote. Hence the curve lies to the right of the asymptote at each end.

If we had written

$$x=my+c+\alpha/y+\beta/y^2+\dots,$$

we should have obtained another solution

$$x=-y-3/y+\dots,$$

which is another form for the branch

$$y=-x+3/x+\dots,$$

which we found before. The reader should verify that both forms give the same situation of the curve relative to the asymptote.

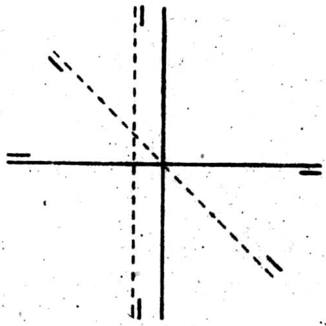


FIG. 1

(ii) $x(x+y)^2-x(x+y)+1=0.$

The directions of the asymptotes are given by $x=0$, $x+y=0$ (we shall see later the significance of the repeated factor). We put

$$x=my+c+\alpha/y+\beta/y^2+\dots$$

(this will give all three asymptotes).

Equating to zero the coefficients of powers of y, we have

for y^3 : $m(m+1)^2=0$. This gives the directions again.

for y^2 : $2cm(m+1)+c(m+1)^2-m(m+1)=0$. If $m=0$, $c=0$: if $m=-1$, the equation is satisfied identically.

for y : $mc^2+2m(m+1)\alpha+2c^2(m+1)+\alpha(m+1)^2-c(2m+1)=0$. If $m=0$, $\alpha=0$: if $m=-1$, $c=0$ or 1.

for y^0 : $2mc\alpha+2m(m+1)\beta+c^3+4c(m+1)\alpha+\beta(m+1)^2-m\alpha-c^2-\alpha(m+1)+1=0$. If $m=0$, $\beta=-1$: if $m=-1$ and $c=0$, $\alpha=-1$: if $m=-1$ and $c=1$, $\alpha=1$.

Hence

$$x=-1/y^2+\dots \text{ or } x=-y-1/y+\dots \text{ or } x=-y+1+1/y+\dots$$

The asymptotes are $x=0$, $x+y=0$, $x+y-1=0$, and we find on which side of the asymptotes the curve lies as before. We now see that there are two asymptotes in the direction $x+y=0$, which accounts for the repeated factor. The curve has a double point at infinity in this direction, which is a node, because the tangents there are distinct. (See fig. 2.)

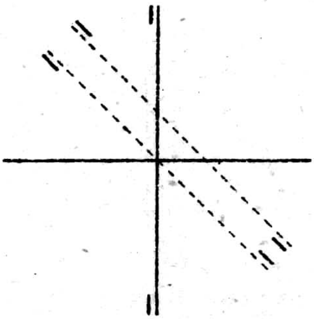


FIG. 2

(iii) $(x+y)(x-y)^2=3xy-y^2$.

The directions of the asymptotes are given by $x+y=0$, $x-y=0$: again we have a repeated factor. We put

$$y=mx+c+\alpha/x+\dots$$

From now onwards, we omit the details and just quote the results. We have

$$(m+1)(m-1)^2=0, \text{ giving } m=-1 \text{ or } 1.$$
$$2c(m^2-1)+c(m-1)^2=3m-m^2.$$

If $m=-1$, $c=-1$: if $m=1$, we obtain a contradiction. Hence this method breaks down for the direction $x-y=0$. The reason is, as we shall see later, that the curve has a non-linear branch. If $m=-1$, we may complete the working as before.

(iv) $x^2y^2+xy+x+3y+3=0$.

The directions of the asymptotes are given by $x=0$ and $y=0$, both repeated. We put

$$y=c+\alpha/x+\dots$$

We have
$$c^2=0,$$
$$2c\alpha+c+1=0,$$

which give a contradiction. The reason why this method breaks down is, as we shall see later, that the curve has branches going to the same end of the asymptote.

(2) The second method which we shall describe is shorter than the first, and has the advantage that it deals successfully with the cases where the first method breaks down. However, it needs very great care in interpreting the results.

Case I. The curve has no double point at infinity.

If we can express the equation as a product of n linear factors