Edited by Ben-yu Guo Zhong-ci Shi

Proceedings of
The Fourth International Workshop on Scientific Computing and Applications

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Preface

This volume is a collection of 18 refereed papers presented at The Fourth International Workshop on Scientific Computing and Applications held on June 20-23, 2005 at Shanghai Jiaotong University, China, co-organized with Shanghai E-Institute for Computational Science. The former workshops of the series were held at City University of Hong Kong, in December 1998 and October 2003, and Banff, Alberta, Canada in May 2000, respectively.

The aim of this series of workshops is to bring together mathematicians, applied scientists and engineers working in the field of scientific computing and its applications, and to provide a forum for the participants to meet and exchange their ideas and experiences in the research work.

In this year, more than one hundred people from different countries attended the four day event. The workshop includes 11 plenary talks and a number of contributed talks, which cover a wide range of topics in the field of scientific computing and its applications.

This workshop was supported by Shanghai Jiaotong University and Shanghai E-Institute for Computational Science. The editors wish to thank Department of Mathematics of Shanghai Jiaotong University for its contributions to the success of the workshop. We have also received significant help from members of the Scientific Committee and Organizing Committee. We are grateful to all authors for their contributions, to referees for reviewing the papers, and to Professor Tian Hong-jiong for preparing this volume.

October, 2005

Ben-yu Guo
Shanghai Normal University
Zhong-ci Shi
Chinese Academy of Sciences

Contents

Recovery of Boundaries and the Boundary Conditions for Multiple Obstacles from the Far-field Pattern

Jin Cheng [a] Jijun Liu [b] Gen Nakamura [c]

[a] *Department of Mathematics, Fudan University*
Shanghai, 200433, P.R.China
E-mail: jcheng@fudan.edu.cn
[b] *Department of Mathematics, Southeast University*
Nanjing, 210096, P.R.China
E-mail: jjliu@seu.edu.cn
[c] *Department of Mathematics, Graduate School of Science*
Hokkaido University, Sapporo, 060-0810, Japan
E-mail: gnaka@math.sci.hokudai.ac.jp

We consider an inverse scattering problem for multiple obstacles $D = \cup_{j=1}^{N} D_j \subset R^3$ with different types of boundary for D_j. By constructing an indicator function from the far-field pattern of scattered wave, we can firstly reconstruct the shape of all obstacles, then identify the type of boundary for each obstacle, as well as the boundary impedance in the case obstacles have Robin-type boundary condition. The novelty of our probe method comparing with the existing probe method is that we succeeded in identifying the type of boundary condition for multiple obstacles by analyzing the behavior of both the imaginary part and the real part of the indicator function.

Keywords: Inverse scattering, probe method, uniqueness, indicator
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1. Introduction

Let D be a bounded domain in R^3 such that $D = \cup_{j=1}^{N} D_j$, $\overline{D_i} \cap \overline{D_j} = \emptyset$ $(i \neq j)$. Each D_j is a simply connected domain with C^2 boundary ∂D_j. The scattering of time-harmonic acoustic plane waves by the obstacle D with some boundary is modelled as an exterior boundary value problem for the Helmholtz equation. That is, for a given incident plane wave $u^i(x) = e^{ikx\cdot d}$, $d \in S^2 = \{\xi \in R^3 : |\xi| = 1\}$, the total wave field $u = u^i + u^s \in H^1_{loc}(R^3 \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } R^3 \setminus \overline{D} \\ Bu(x,t) = 0, & \text{on } \partial D \\ \dfrac{\partial u^s}{\partial r} - iku^s = O\left(\dfrac{1}{r}\right), r = |x| \longrightarrow \infty \end{cases} \tag{1.1}$$

where B is a boundary operator corresponding to different types of the obstacle D, that is,

$$Bu = \begin{cases} u & \text{if } \partial D_j \text{ is sound-soft,} \\ \dfrac{\partial u}{\partial \nu} & \text{if } \partial D_j \text{ is sound-hard,} \\ \dfrac{\partial u}{\partial \nu} + i\sigma(x)u & \text{if } \partial D_j \text{ is Robin-type,} \end{cases} \tag{1.2}$$

where ν is the unit normal on ∂D directed into the exterior of D, $\sigma(x) > 0$ is the boundary impedance coefficient. By the results in [5], we know that there exists a unique solution for the forward scattering problem (1.1).

For the incident field $u^i(x) = e^{ikx\cdot d}$, the far-field pattern $u^\infty(\theta, d)$ can be defined by

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u^\infty(\theta,d) + O\left(\frac{1}{|x|}\right)\right\}, \qquad |x| \longrightarrow \infty,$$

where $\theta, d \in S^2$.

Generally, the inverse scattering problem corresponding to (1.1) is to identify the boundary ∂D and also $\sigma(x)$ in case of Robin-type boundary, from a knowledge of far-field pattern. If D is just one obstacle, then identifying ∂D for each kind of boundary conditions has been discussed thoroughly. For example, if D is sound-soft (Dirichlet boundary condition on ∂D) or sound-hard (Neumann boundary condition on ∂D), the problems have been studied by many researchers, see [3], [6], [8], [9], [11], [14], [18]. In the case of obstacle with Robin-type boundary, the problem of reconstructing $\sigma(x)$, when ∂D is given, has also been studied , see [4], [6], [16], [17]. For the inverse scattering problem of determining both ∂D and boundary impedance, an approximate determination (or reconstruction) of the shape of D and boundary impedance was discussed in [20] by using the asymptotic behavior of the low frequency scattered waves associated with three different incident waves (or frequencies). In [13], one numerical method is proposed to determine both ∂D and impedance $\sigma(x)$. In [1], the authors gave a uniqueness and reconstruction formula for identifying ∂D and the impedance for a Robin-type obstacle from the far-field pattern, by applying the probe method introduced by M. Ikehata (see [8], [9], [10], [11] and [12] for example). Moreover, it has also been noticed that the probe method, as well as the point-source method proposed in [19], can be applied to determine the boundaries of multiple obstacles, if their boundary types are the same (sound-soft or sound-hard). Now, we propose a new problem: if there are many obstacles with different types of boundary such as sound-soft, sound-hard, as well as Robin-type, can we still identify their shapes and locations as well as the type of boundary for each obstacle?

This is the main topic of this paper. Our answer to this problem is "yes". More precisely, our result can be stated as follows.

Theorem 1.1. Let D be a bounded domain consisting of finite obstacles D_j $(j = 1, 2, \cdots, N)$, namely, $D = \cup_{j=1}^{N} D_j$. We assume that each obstacle D_j is simply connected bounded domain with C^2 boundary ∂D_j and $\overline{D}_i \cap \overline{D}_j = \emptyset$ for $i \neq j$. For given incident plane waves $u^i(x, d) = e^{ikx \cdot d}$, consider the following scattering problem for total wave field $u(x, d) = u^i(x, d) + u^s(x, d)$:

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } R^3 \setminus \overline{D} \\ B_j u(x,t) = 0, & \text{on } \partial D_j, j = 1, 2, \cdots, N \\ \dfrac{\partial u^s}{\partial r} - iku^s = O\left(\dfrac{1}{r}\right), r = |x| \longrightarrow \infty, \end{cases} \tag{1.3}$$

where B_j is one of the boundary operator in (1.2) for $j = 1, 2, \cdots, N$. Assume that $\overline{D} \subset \Omega$ for some known sphere Ω and $0 < \sigma_j(x) \in C(\partial D_j)$ for Robin-type obstacles D_j. If there exists at least one Robin-type obstacle, then from the far-field pattern $u^\infty(\theta, d)$ for all $\theta, d \in S^2$, we can

(1) determine the number of obstacles N,
(2) reconstruct ∂D_j for $j = 1, 2, \cdots, N$,
(3) identify the type of each obstacle D_j,
(4) reconstruct $\sigma_j(x)$ for the Robin-typed obstacles D_j.

Our main tool to deal this problem is still the probe method. This method gives a reconstruction formula for the shape and location of an obstacle by using the indicator function and analyzing its behavior. However, there are some new ingredients in this paper. In the case of multiple obstacles, we not only have to determine the shape and location of each obstacle, but also we have to determine the number of obstacles and identify the type of each obstacle. This is the major and important difference between the multiple-obstacle inversion and single-obstacle one. Especially, we have to extract some characteristics of the indicator function such that we can distinguish sound-hard obstacles and obstacles with Robin-type boundary, since in most cases, we can consider the Neumann boundary as the special case of Robin boundary with $\sigma(x) = 0$. Then the most important ingredient of this paper is that we succeeded in providing a method distinguishing sound-hard boundary and Robin-type boundary. More precisely, we can determine the positions of obstacles and identify sound-soft boundary from the real part of the indicator function, while distinguishing the sound-hard boundary from Robin-type boundary is done by considering the imaginary part of the indicator function. In order to carry out this, the most important and difficult thing is to rewrite the indicator function in an appropriate form and analyze its behavior. The number of obstacles can be obtained immediately when we get the whole image of all obstacles.

We will give a mathematically rigorous reconstruction formula for recovering ∂D_j for $j = 1, 2, \cdots, N$. Then the uniqueness of identifying ∂D_j and the determination of number of obstacles from $u^\infty(d, \theta)$ for all $d, \theta \in S^2$ becomes obvious from the reconstruction. Since our reconstruction procedure is point wise, it is enough to consider the case that D consists of 3 obstacles with sound-soft, sound-hard and Robin-type boundary respectively, and to illustrate the reconstruction procedure for identifying the location and type for each obstacle. This does not loose any generality. More precisely, we assume that D_1, D_2, D_3 are sound soft, sound hard and Robin-type, respectively. Once we have identified the shape, location and type of each obstacle, we determine $\sigma(x) := \sigma_3(x)$ on ∂D_3 by the moment method. So, henceforth we assume $N = 3$.

Our paper is organized as follows:

- Section 2: Preliminary results
- Section 3: Probe method
- Section 4: Moment method for determining $\sigma(x)$
- Section 5: Some estimates
- Section 6: Singularity Analysis

2. Preliminary results

In this section, without the proofs, we give some known results for the probe method, which are necessary for our paper.

Without loss of generality, we assume that $\overline{D} \subset B(0, \frac{R}{2})$ for some constant $R > 0$. We also assume that 0 is not a Dirichlet eigenvalue of $\Delta + k^2$ in $\Omega := B(0, R)$ for given $k > 0$.

Proposition 2.1. The scattered solution $u^s(x, d)$ for $|x| > \frac{R}{2}$ can be determined uniquely from $u^\infty(d, \theta)$.

Let $G(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$ be the fundamental solution of the Helmholtz equation. For each $y \in R^3 \setminus \overline{D}$, we define $E(\cdot, y) \in H^1_{loc}(R^3 \setminus \overline{D})$ as the solution to

$$\begin{cases} \Delta E + k^2 E = 0, & \text{in } R^3 \setminus \overline{D} \\ B_i E(x, y) = -B_i G(x, y), & \text{on } \partial D_i, \quad i = 1, 2, 3 \\ \dfrac{\partial E}{\partial r} - ikE = O\left(\dfrac{1}{r}\right), & r = |x| \longrightarrow \infty. \end{cases} \tag{2.1}$$

Proposition 2.2. For $x, y \in \partial\Omega$, $E(x,y)$, $\frac{\partial}{\partial\nu(x)}E(x,y)$ and $\frac{\partial}{\partial\nu(y)}E(x,y)$ can be determined from $u^\infty(d,\theta)$ for all $d, \theta \in S^2$.

The proof for $D = D_1 \cup D_2 \cup D_3$ given here is an analogy to that given in [1] for $D = D_3$.

Consider a solution $u(x) \in H^1(\Omega \setminus \overline{D})$ to the following boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \setminus \overline{D} \\ B_j u(x,t) = 0, & \text{on } \partial D_j, j = 1,2,3 \\ u(x) = f, & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

for given $f \in H^{1/2}(\partial\Omega)$.

Since we have used $\overline{D}$ to indicate the closure of domain D, we will use $\tilde{z}$ to indicate the complex conjugate of complex number z in the sequel.

Lemma 2.3. If $D_3 \neq \emptyset$, then there exists a unique solution to (2.2) for any $f \in H^{1/2}(\partial\Omega)$.

Define the Dirichlet-to-Neumann map $\Lambda_{\partial D,\sigma}$ formally by

$$\Lambda_{\partial D,\sigma} : f \longrightarrow \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega),$$

where $u \in H^1(\Omega \setminus \overline{D})$ is the solution of (2.2) for $f \in H^{1/2}(\partial\Omega)$. In the next Lemma, we show the relations between the far-field patterns and the Dirichlet-to-Neumann map.

Lemma 2.4. Let u be the solution to (2.2) for $f \in H^{1/2}(\partial\Omega)$. Then, $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ can be obtained from $f(x)$ and $u^\infty(d,\theta)$ for $\theta, d \in S^2$.

From this lemma, we see that the original inverse problem can be restated as the problem of reconstructing the shapes of the 3 obstacles and the boundary impedance of D_3 from the Dirichlet-to-Neumann map $\Lambda_{\partial D,\sigma}$.

Corresponding to the case $D = \emptyset$, we can formally define the Dirichlet-to-Neumann map $\Lambda_{0,0} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ by

$$\Lambda_{0,0} : f \longrightarrow \frac{\partial u_1}{\partial \nu}|_{\partial\Omega},$$

where $u_1(x) \in H^1(\Omega)$ is the solution to

$$\begin{cases} \Delta u_1 + k^2 u_1 = 0, & \text{in } \Omega \\ u_1(x) = f \in H^{1/2}(\partial\Omega), & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here note that by the assumption 0 is not the Dirichlet eigenvalue of the operator $\Delta + k^2$ in Ω, (2.3) is uniquely solvable.

The weak formula of $\Lambda_{0,0}$ is given by

$$\langle \Lambda_{0,0} f, g \rangle = \int_\Omega (\nabla u_1 \cdot \nabla v - k^2 u_1 v) dx, \tag{2.4}$$

where u_1 is the solution of (2.3) for $f \in H^{1/2}(\partial\Omega)$ and $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = g$ for $g \in H^1(\partial\Omega)$. For the solution u of (2.2) and the solution u_1 of (2.3), we have

Lemma 2.5. Let $u \in H^1(\Omega \setminus \overline{D})$ and $u_1 \in H^1(\Omega)$ be the solutions to (2.2) and (2.3), respectively. There exists a constant $C = C(k, R, \sigma_0)$ such that, for all $f \in H^{1/2}(\partial\Omega)$,

$$\|u - u_1\|_{H^1(\Omega\setminus\overline{D})} \leqslant C \|u_1\|_{H^1(D)},$$

where $\sigma_0 > 0$ is a constant satisfying $0 < \sigma(x) \leqslant \sigma_0$, $x \in \partial D_3$.

The proof of Lemma 2.5 is almost the same as that given in [10].

3. Probe Method

Definition 3.1. For any non-self-intersecting continuous curve $c = \{c(t)\,;\, 0 \leqslant t \leqslant 1\}$, if it satisfies $c(0), c(1) \in \partial\Omega$ and $c(t) \in \Omega$ $(0 < t < 1)$, then we call c a needle in Ω.

Definition 3.2. For any needle c in Ω, we call

$$t(c, D) = \sup\{0 < t < 1;\ c(s) \in \Omega \setminus \overline{D} \text{ for all } 0 < s < t\}$$

geometric impact parameter(GIP). It is obvious that $t(c, D) = 1$ if c does not touch any point on ∂D.

From this definition, we know if a needle c touches $\overline{D}$, then $t(c, D) < 1$ and $t(c, D)$ is the first hitting time, i.e., $c(t(c, D)) \in \partial D$ and $c(t) \in \Omega \setminus \overline{D}$ for $0 < t < t(c, D)$ if we consider t as a time.

Since $\Omega \setminus \overline{D}$ is connected, we have a reconstruction algorithm for ∂D in terms of the geometric impact parameter and the needle, i.e.,

$$\partial D = \{c(t)\,;\, t = t(c, D),\ c \text{ is a needle and } t(c, D) < 1\}. \tag{3.1}$$

Therefore, in order to reconstruct ∂D, it suffices to consider the problem of calculating the GIP for each needle from the Dirichlet-to-Neumann map.

Lemma 3.3. Suppose that Γ is an arbitrary open set of $\partial\Omega$. For each $0 < t < 1$, there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in $H^1(\Omega)$, which satisfies

$$\Delta v_n + k^2 v_n = 0$$

such that $supp(v_n|_{\partial\Omega}) \subset \Gamma$ and

$$v_n \longrightarrow G(\cdot - c(t)) \quad \text{in} \quad H^1_{loc}(\Omega \setminus c_t),$$

where $c_t := \{c(t');\ 0 < t' \leqslant t\}$.

This result comes from the Runge approximation theorem, see [8], [9].

Remark 3.4. Usually the Runge approximation is not constructive, because its proof is done by using the unique continuation and Hahn-Banach theorem. However, for the Helmholtz equation, it is possible to make the Runge approximation constructive by using the translation theory (see [7]).

It is obvious that $v_n|_{\partial\Omega}$ depends on $c(t)$. We denote it by $v_n|_{\partial\Omega} = f_n(\cdot, c(t))$, where $f_n(\cdot, c(t)) \in H^{1/2}(\partial\Omega)$ and $supp(f_n(\cdot, c(t))) \subset \Gamma$.

Definition 3.5. For a given needle c in Ω and $0 < t < 1$, we define the indicator function

$$I(t,c) = \lim_{n\to\infty} \langle \overline{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t))}, f_n(\cdot, c(t))\rangle \tag{3.2}$$

whenever it is defined, where $\langle \cdot,\cdot \rangle$ is the pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, $\overline{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t))}$ is the complex conjugate of $(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t))$.

Next we show that $\Re I(t,c)$ and $\Im I(t,c)$ ($\Re, \Im$ denote the real part and imaginary part respectively) can be used to calculate GIP from which the shapes and locations of 3 obstacles can be determined, and we can also identify the type of each obstacle.

Theorem 3.6. For a given needle $c(t)$ in Ω, it follows that

(A) $t(c, D) = 1$ if and only if $I(t,c)$ exists for all $0 \leqslant t \leqslant 1$ and

$$\sup_{0\leqslant t\leqslant 1} |\Re I(t,c)| < +\infty.$$

(B) $T = t(c, D)$ can be characterized by

(1) $I(t,c)$ exists for all $0 \leqslant t < T$ and

$$\sup_{0\leqslant s\leqslant t} |\Re(I(s,c))| < +\infty, \qquad \textit{for} \quad 0 \leqslant t < T,$$

(2) $\lim_{t\to T-} |\Re I(t,c)| = +\infty$.

(C) We can identify ∂D_i for $i = 1, 2, 3$ by

$$\lim_{t\to t(c,D)-} \Re I(t,c) = +\infty \iff c(t(c,D)) \in \partial D_1,$$

$$\lim_{t\to t(c,D)-} \Re I(t,c) = -\infty \text{ and } \lim_{t\to t(c,D)-} \Im I(t,c) < +\infty \iff c(t(c,D)) \in \partial D_2,$$

$$\lim_{t\to t(c,D)-} \Re I(t,c) = -\infty \text{ and } \lim_{t\to t(c,D)-} \Im I(t,c) = +\infty \iff c(t(c,D)) \in \partial D_3.$$

Remark 3.7. The result (A), (B) give a criterion for the geometric impact parameter $t(c,D)$ in terms of the indicator function. It is easy to see that T with the properties (1), (2) is given by

$$T = \sup\{0 < t < 1;\ \sup_{0\leqslant s\leqslant t} |\Re I(s,c)| < +\infty\}.$$

Furthermore, since $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$, we can identify ∂D_i according to (C).

Proof. For a given needle $c(t)$, by Lemma 3.3, we know that there exists a sequences $\{v_n(x)\} \in H^1(\Omega)$ which satisfies

$$\begin{cases} \Delta v_n + k^2 v_n = 0, \text{ in } \Omega \\ v_n = f_n(\cdot, c(t)), \quad \text{on } \partial\Omega\ ; \qquad supp(f_n(\cdot, c(t))) \subset \Gamma, \end{cases}$$

and

$$v_n \longrightarrow G(\cdot - c(t)) \qquad \text{in} \quad H^1_{loc}(\Omega \setminus c_t) \quad (n \longrightarrow \infty).$$

Let $u_n(x) \in H^1(\Omega \setminus \overline{D})$ satisfy

$$\begin{cases} \Delta u_n + k^2 u_n = 0, \text{ in } \Omega \setminus \overline{D} \\ B_i u_n = 0, \qquad \text{on } \partial D_i, \quad i = 1, 2, 3 \\ u_n = f_n, \qquad \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

then $w_n = u_n - v_n|_{\Omega\setminus\overline{D}} \in H^1(\Omega \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta w_n + k^2 w_n = 0, \text{ in } \Omega \setminus \overline{D} \\ B_j w_n = -B_j v_n, \quad \text{on } \partial D_j, \quad j = 1, 2, 3 \\ w_n = 0, \qquad \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.5 and Lemma 3.3, we know that, for $c_t \cap \overline{D} = \emptyset$, it holds that

$$w_n \longrightarrow w \quad \text{in} \quad H^1(\Omega \setminus \overline{D}), \qquad n \longrightarrow \infty, \tag{3.4}$$

where $w = w(x, c(t))$ satisfies

$$\begin{cases} \Delta w + k^2 w = 0, & \text{in } \Omega \setminus \overline{D} \\ B_j w = -B_j G(\cdot - c(t)), & \text{on } \partial D_j, \quad j = 1,2,3 \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

On the other hand, by the calculation in Section 6, we have two kinds of expressions for $\widetilde{\langle(\Lambda_{\partial D,\sigma} - \Lambda_{0,0})f_n}, f_n\rangle$, i.e.,

$$\begin{aligned} &\widetilde{\langle(\Lambda_{\partial D,\sigma} - \Lambda_{0,0})f_n(\cdot, c(t))}, f_n(\cdot, c(t))\rangle \\ &= -\int_{\Omega\setminus\overline{D}} (|\nabla w_n|^2 - k^2|w_n|^2)dx - \int_D (|\nabla v_n|^2 - k^2|v_n|^2)dx \\ &\quad + \int_{\partial D_3} \left(2i\sigma v_n \tilde{w}_n + i\sigma|v_n|^2 + i\sigma|w_n|^2\right) ds \\ &\quad + \int_{\partial D_1} \left[\left(v_n \frac{\partial \tilde{w}_n}{\partial\nu} - \tilde{w}_n \frac{\partial v_n}{\partial\nu}\right) + \left(v_n \frac{\partial \tilde{v}_n}{\partial\nu} - \tilde{w}_n \frac{\partial w_n}{\partial\nu}\right)\right] ds. \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\widetilde{\langle(\Lambda_{\partial D,\sigma} - \Lambda_{0,0})f_n(\cdot, c(t))}, f_n(\cdot, c(t))\rangle \\ &= \int_{\Omega\setminus\overline{D}} \left(|\nabla w_n|^2 - k^2|w_n|^2\right) dx + \int_D \left(|\nabla v_n|^2 - k^2|v_n|^2\right) dx \\ &\quad - \int_{\partial D_2} \left[\tilde{u}_n \frac{\partial v_n}{\partial\nu} + u_n \frac{\partial \tilde{v}_n}{\partial\nu}\right] ds - \int_{\partial D_3} \left[\tilde{u}_n \frac{\partial v_n}{\partial\nu} + u_n \frac{\partial \tilde{v}_n}{\partial\nu} - i\sigma|u_n|^2\right] ds \end{aligned} \tag{3.7}$$

Let n tend to infinity in (3.6). Then, by (3.4), we have

$$\begin{aligned} -I(t,c) = &\int_D \{|\nabla G(\cdot - c(t))|^2 - k^2|G(\cdot - c(t))|^2\}dx \\ &+ \int_{\Omega\setminus\overline{D}} \{|\nabla w|^2 - k^2|w|^2\}dx \\ &- i\int_{\partial D_3} \sigma(x)\{|G(\cdot - c(t))|^2 + |w|^2\}ds - 2i\int_{\partial D_3} \sigma(x)\tilde{w}G ds \\ &- \int_{\partial D_1} \left[\left(G\frac{\partial\tilde{w}}{\partial\nu} - \tilde{w}\frac{\partial G}{\partial\nu}\right) + \left(G\frac{\partial\tilde{G}}{\partial\nu} - \tilde{w}\frac{\partial w}{\partial\nu}\right)\right] ds. \end{aligned} \tag{3.8}$$

We first prove (A). If $t(c, D) = 1$, then, by the definition of $t(c, D)$, we know that $c(t)$ does not touch $\overline{D}$, i.e. $c(t) \in \Omega \setminus \overline{D}$ for $0 \leqslant t \leqslant 1$. It is easy to verify that

$$\sup_{0\leqslant t\leqslant 1} |\Re I(t,c)| < +\infty.$$

The converse is true if we prove (B). So, let's go to the proof of (B). Let $t(c,D) < 1$ and $x_0 = c(t(c,D)) \in \partial D$. Then, it is easy to see that we only have to prove $\lim_{t\to t(c,D)-} |\Re I(t,c)| = +\infty$. This is included in the proof of (C). So the rest of the argument is devoted to the proof of (C).

Likewise before, since $c(t) \in \Omega \setminus \overline{D}$ $(0 \leqslant t < t(c,D))$, we have $\sup_{0\leqslant s\leqslant t}|\Re I(s,c)| < +\infty$ for $0 \leqslant t < t(c,D)$. Now observe that

$$\begin{aligned}
-\Re I(t,c) =& \int_D [|\nabla G(\cdot - c(t))|^2 - k^2|G(\cdot - c(t))|^2]dx \\
&+ \int_{\Omega\setminus\overline{D}} (|\nabla w|^2 - k^2|w|^2)dx + 2\int_{\partial D_3} \sigma(x)\Im(\tilde{w}G(\cdot - c(t)))ds \\
&-\Re\left[\int_{\partial D_1}\left(G\frac{\partial\tilde{w}}{\partial\nu} - \tilde{w}\frac{\partial G}{\partial\nu} + G\frac{\partial\tilde{G}}{\partial\nu} - \tilde{w}\frac{\partial w}{\partial\nu}\right)ds\right] \\
\geqslant& \int_D |\nabla G(\cdot - c(t))|^2dx - k^2\int_D |G(\cdot - c(t))|^2dx - k^2\int_{\Omega\setminus\overline{D}} |w|^2dx \\
&+2\int_{\partial D_3} \sigma(x)\Im(\tilde{w}G(\cdot - c(t)))ds \\
&-\Re\left[\int_{\partial D_1}\left(G\frac{\partial\tilde{w}}{\partial\nu} - \tilde{w}\frac{\partial G}{\partial\nu} + G\frac{\partial\tilde{G}}{\partial\nu} - \tilde{w}\frac{\partial w}{\partial\nu}\right)ds\right]. \qquad (3.9)
\end{aligned}$$

According to the result of singularity analysis about $w(x,x_0)$ and $G(x-x_0)$ for $x_0 \in \partial D$, we have from (3.9),

$$\lim_{t\to t(c,D_j)-} \Re(I(t,c)) = -\infty,$$

if $j = 2,3$. On the other hand, consider the real part of the limit of (3.7) as $n \to \infty$. It is easy to find that the real part will tend to $+\infty$ when $c(t) \to \partial D_1$, since $\int_D |\nabla G(\cdot - c(t))|^2dx$ will blow up, while the integrals on the boundary are clearly bounded and L^2 integral of w is bounded. These facts imply that we can distinguish the sound-soft boundary D_1 from the other two kinds of boundaries (sound-hard and Robin-type). Now we want to distinguish ∂D_2 and ∂D_3 furthermore. For this purpose, we need to consider the imaginary part of (3.7). In fact, it yields from (3.6) that

$$\Im(\langle\widetilde{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0})f_n}, f_n\rangle) = \int_{\partial D_3} \sigma(x)|u_n|^2ds. \qquad (3.10)$$

Now we estimate the behavior of the imaginary part of indicator function. Remind our previous notations, we get

$$\lim_{n\to\infty} \Im(\langle\widetilde{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0})f_n}, f_n\rangle)$$

$$= \lim_{n\to\infty} \int_{\partial D_3} \sigma(x)|u_n|^2 ds$$
$$= \lim_{n\to\infty} \int_{\partial D_3} \sigma(x)|v_n + w_n|^2 ds$$
$$= \int_{\partial D_3} \sigma(x)|(G(x-c(t))) + w(x, c(t))|^2 ds, \tag{3.11}$$

where w is the function defined by (3.5). According to the singularity analysis in section 5 and section 6, we know that $|G(x-c(t)) + w(x, c(t))|$ is estimated by $|G(x-c(t))|$. Hence, from (3.11) and the estimate for G in section 5, we have

$$\lim_{t\to t(c,D_3)-} \Im I(t,c) = \lim_{t\to t(c,D_3)-} \int_{\partial D_3} \sigma(x)|(G(x-c(t))) + w(x,c(t))|^2 ds = +\infty,$$

$$\lim_{t\to t(c,D_2)-} \Im I(t,c) = \lim_{t\to t(c,D_2)-} \int_{\partial D_3} \sigma(x)|G(x-c(t)) + w(x,c(t))|^2 ds < \infty.$$

Since D_2 and D_3 are separated, these behavior of $\Im I(t,c)$ enable us to distinguish ∂D_3 and ∂D_2.

The proof is complete. □

Now we give the reconstruction procedure for the shape and type of each obstacle. It can be realized by the following steps:

- Calculate the Dirichlet-to-Neumann map $\Lambda_{\sigma,D}$ from the far field patterns $u^\infty(d,\theta),\ d,\theta \in S^2$.
- For any given needle $c(t)$, calculate the sequences v_n and $f_n(\cdot, c)$.
- Calculate $\langle (\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t)), f_n(\cdot, c(t))\rangle$.
- Calculate $I(t,c)$ and

$$\partial D_1 = \{c(t_0);\ \sup_{0\leqslant s\leqslant t}|\Re I(s,c)| < +\infty \quad \text{for} \quad 0 \leqslant t < t_0,\ \lim_{t\to t_0-} \Re I(t,c) = +\infty\},$$

$$\partial D_2 = \{c(t_0);\ \sup_{0\leqslant s\leqslant t}|\Re I(s,c)|, \sup_{0\leqslant s\leqslant t}|\Im I(s,c)| < +\infty \text{ for } 0\leqslant t<t_0,\ \lim_{t\to t_0-} \Re I(t,c) = -\infty \text{ and } \lim_{t\to t_0-} \Im I(t,c) < +\infty\},$$

$$\partial D_3 = \{c(t_0);\ \sup_{0\leqslant s\leqslant t}|\Re I(s,c)|, \sup_{0\leqslant s\leqslant t}|\Im I(s,c)| < +\infty \text{ for } 0\leqslant t<t_0,\ \lim_{t\to t_0-} \Re I(t,c) = -\infty \text{ and } \lim_{t\to t_0-} \Im I(t,c) = +\infty\}.$$

The rest of the proof of Theorem 1.1 is to reconstruct boundary impedance on D_3. This will be given in the next section.