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Kunihiko Kodaira

Complex Manifolds and Deformation of Complex Structures

复流形和复结构的形变

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Kunihiko Kodaira

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**Dedicated to my esteemed colleague and friend
D. C. Spencer**

Preface

This book is an introduction to the theory of complex manifolds and their deformations.

Deformation of the complex structure of Riemann surfaces is an idea which goes back to Riemann who, in his famous memoir on Abelian functions published in 1857, calculated the number of effective parameters on which the deformation depends. Since the publication of Riemann's memoir, questions concerning the deformation of the complex structure of Riemann surfaces have never lost their interest.

The deformation of algebraic surfaces seems to have been considered first by Max Noether in 1888 (M. Noether: *Anzahl der Modulen einer Classe algebraischer Flächen*, Sitz. Königlich. Preuss. Akad. der Wiss. zu Berlin, erster Halbband, 1888, pp. 123–127). However, the deformation of higher dimensional complex manifolds had been curiously neglected for 100 years. In 1957, exactly 100 years after Riemann's memoir, Frölicher and Nijenhuis published a paper in which they studied deformation of higher dimensional complex manifolds by a differential geometric method and obtained an important result. (A. Frölicher and A. Nijenhuis: A theorem on stability of complex structures, *Proc. Nat. Acad. Sci., U.S.A.*, **43** (1957), 239–241).

Inspired by their result, D. C. Spencer and I conceived a theory of deformation of compact complex manifolds which is based on the primitive idea that, since a compact complex manifold M is composed of a finite number of coordinate neighbourhoods patched together, its deformation would be a shift in the patches. Quite naturally it follows from this idea that an infinitesimal deformation of M should be represented by an element of the cohomology group $H^1(M, \Theta)$ of M with coefficients in the sheaf Θ of germs of holomorphic vector fields. However, there seemed to be no reason that any given element of $H^1(M, \Theta)$ represents an infinitesimal deformation of M . In spite of this, examination of familiar examples of compact complex manifolds M revealed a mysterious phenomenon that $\dim H^1(M, \Theta)$ coincides with the number of effective parameters involved in the definition of M . In order to clarify this mystery, Spencer and I developed the theory of deformation of compact complex manifolds. The process of the development was the most interesting experience in my whole mathematical life. It was similar to an experimental science developed by

the interaction between experiments (examination of examples) and theory. In this book I have tried to reproduce this interesting experience; however I could not fully convey it. Such an experience may be a passing phenomenon which cannot be reproduced.

The theory of deformation of compact complex manifolds is based on the theory of elliptic partial differential operators expounded in the Appendix. I would like to express my deep appreciation to Professor D. Fujiwara who kindly wrote the Appendix and also to Professor K. Akao who spent the time and effort translating this book into English.

Tokyo, Japan
January, 1985

KUNIHICO KODAIRA

Contents

CHAPTER 1	
Holomorphic Functions	1
§1.1. Holomorphic Functions	1
§1.2. Holomorphic Map	23
CHAPTER 2	
Complex Manifolds	28
§2.1. Complex Manifolds	28
§2.2. Compact Complex Manifolds	39
§2.3. Complex Analytic Family	59
CHAPTER 3	
Differential Forms, Vector Bundles, Sheaves	76
§3.1. Differential Forms	76
§3.2. Vector Bundles	94
§3.3. Sheaves and Cohomology	109
§3.4. de Rham's Theorem and Dolbeault's Theorem	134
§3.5. Harmonic Differential Forms	144
§3.6. Complex Line Bundles	165
CHAPTER 4	
Infinitesimal Deformation	182
§4.1. Differentiable Family	182
§4.2. Infinitesimal Deformation	188
CHAPTER 5	
Theorem of Existence	209
§5.1. Obstructions	209
§5.2. Number of Moduli	215
§5.3. Theorem of Existence	248
CHAPTER 6	
Theorem of Completeness	284
§6.1. Theorem of Completeness	284
§6.2. Number of Moduli	305
§6.3. Later Developments	314

CHAPTER 7	
Theorem of Stability	320
§7.1. Differentiable Family of Strongly Elliptic Differential Operators	320
§7.2. Differentiable Family of Compact Complex Manifolds	345
APPENDIX	
Elliptic Partial Differential Operators on a Manifold	363
by Daisuke Fujiwara	
§1. Distributions on a Torus	363
§2. Elliptic Partial Differential Operators on a Torus	391
§3. Function Space of Sections of a Vector Bundle	419
§4. Elliptic Linear Partial Differential Operators	430
§5. The Existence of Weak Solutions of a Strongly Elliptic Partial Differential Equation	438
§6. Regularity of Weak Solutions of Elliptic Linear Partial Differential Equations	443
§7. Elliptic Operators in the Hilbert Space $L^2(X, B)$	445
§8. C^∞ Differentiability of $\varphi(t)$	452
Bibliography	459
Index	461

Holomorphic Functions

§1.1. Holomorphic Functions

(a) Holomorphic Functions

We begin by defining holomorphic functions of n complex variables. The n -dimensional complex number space is the set of all n -tuples (z_1, \dots, z_n) of complex numbers $z_i, i = 1, \dots, n$, denoted by \mathbb{C}^n . \mathbb{C}^n is the Cartesian product of n copies of the complex plane: $\mathbb{C}^n = \mathbb{C} \times \dots \times \mathbb{C}$. Denoting (z_1, \dots, z_n) by z , we call $z = (z_1, \dots, z_n)$ a point of \mathbb{C}^n , and z_1, \dots, z_n the complex coordinates of z . Letting $z_j = x_{2j-1} + ix_{2j}$ by decomposing z_j into its real and imaginary parts (where $i = \sqrt{-1}$), we can express z as

$$z = (x_1, x_2, \dots, x_{2n-1}, x_{2n}). \quad (1.1)$$

Thus \mathbb{C}^n is considered as the $2n$ -dimensional real Euclidean space \mathbb{R}^{2n} equipped with the complex coordinates. $x_1, x_2, \dots, x_{2n-1}, x_{2n}$ are called the real coordinates of z . Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in \mathbb{C}^n . We define the linear combination $\lambda z + \mu w$ of z and w , viewed as vectors, by

$$\lambda z + \mu w = (\lambda z_1 + \mu w_1, \dots, \lambda z_n + \mu w_n),$$

where λ and μ are complex numbers. This makes \mathbb{C}^n a complex linear space. The length of $z = (z_1, \dots, z_n)$ is defined by

$$|z| = \sqrt{|z_1|^2 + \dots + |z_n|^2}. \quad (1.2)$$

Clearly we have

$$|\lambda z| = |\lambda| |z|, \quad (1.3)$$

$$|z + w| \leq |z| + |w|. \quad (1.4)$$

The distance of the two points $z, w \in \mathbb{C}^n$ is given by

$$|z - w| = \sqrt{|z_1 - w_1|^2 + \dots + |z_n - w_n|^2}. \quad (1.5)$$

We introduce a topology on \mathbb{C}^n by the identification with \mathbb{R}^{2n} with the usual topology. Thus, for example, a subset $D \subset \mathbb{C}^n$ is a domain in \mathbb{C}^n if D is a domain considered as a subset of \mathbb{R}^{2n} . Again, a complex-valued function $f(z) = f(z_1, \dots, z_n)$ defined on a subset D in \mathbb{C}^n is continuous if $f(z)$ is so as a function of the real coordinates x_1, x_2, \dots, x_{2n} .

Now we consider a complex-valued function $f(z) = f(z_1, \dots, z_n)$ of n complex variables z_1, \dots, z_n defined on a domain $D \subset \mathbb{C}^n$.

Definition 1.1. If $f(z) = f(z_1, \dots, z_n)$ is continuous in $D \subset \mathbb{C}^n$, and holomorphic in each variable z_k , $k = 1, \dots, n$, separately, $f(z_1, \dots, z_n)$ is said to be *holomorphic* in D . We also call $f(z) = f(z_1, \dots, z_n)$ a *holomorphic function of n variables* z_1, \dots, z_n .

Here, by saying that $f(z_1, \dots, z_k, \dots, z_n)$ is holomorphic in z_k separately, we mean that $f(z_1, \dots, z_n)$ is a holomorphic function in z_k when the other variables $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n$ are fixed.

The fundamental Cauchy integral formula with respect to a circle for holomorphic functions of one variable is extended to the case of holomorphic functions of n variables as follows.

Given a point $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ and positive real numbers r_1, \dots, r_n , we put

$$U_r(c) = \{z | z = (z_1, \dots, z_n) | |z_k - c_k| < r_k, k = 1, \dots, n\}, \quad (1.6)$$

where r denotes (r_1, \dots, r_n) . Let $U_{r_k}(c_k)$ be the disk with centre c_k and radius r_k on the z_k -plane. Then we have

$$U_r(c) = U_{r_1}(c_1) \times \dots \times U_{r_n}(c_n). \quad (1.7)$$

Thus we call $U_r(c)$ the polydisk with centre c . We denote by C_k the boundary of $U_{r_k}(c_k)$, that is, the circle of radius r_k with centre c_k on the z_k -plane. Of course C_k is represented by the usual parametrization $\theta_k \rightarrow \gamma(\theta_k) = c_k + r_k e^{i\theta_k}$ where $0 \leq \theta_k \leq 2\pi$. The product of C_1, C_2, \dots, C_n

$$C^n = C_1 \times \dots \times C_n \quad (1.8)$$

is called the determining set of the polydisk $U_r(c)$. C^n is an n -dimensional torus. Given a continuous function $\psi(\zeta) = \psi(\zeta_1, \dots, \zeta_n)$, with $\zeta_1 \in C_1, \dots, \zeta_n \in C_n$, we define its integral over C^n by

$$\begin{aligned} \int_{C^n} \psi(\zeta) d\zeta_1 \cdots d\zeta_n &= \int_{C_1} \cdots \int_{C_n} \psi(\zeta) d\zeta_1 \cdots d\zeta_n \\ &= \int_0^{2\pi} \cdots \int_0^{2\pi} \psi(\gamma_1(\theta_1), \dots, \gamma_n(\theta_n)) \gamma_1'(\theta_1) \cdots \gamma_n'(\theta_n) d\theta_1 \cdots d\theta_n. \end{aligned} \quad (1.9)$$

Theorem 1.1. Let $f = f(z_1, \dots, z_n)$ be a holomorphic function in a domain $D \subset \mathbb{C}^n$. Take a polydisk $U_r(c)$ with $[U_r(c)] \subset D$. Then for $z \in U_r(c)$, $f(z)$ is represented as

$$f(z) = \left(\frac{1}{2\pi i} \right)^n \int_{C^n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n \quad (1.10)$$

where $[]$ denotes the closure.

Proof. First we consider the case $n = 2$. In this case the right-hand side of (1.10) becomes

$$\left(\frac{1}{2\pi i} \right)^2 \int_0^{2\pi} \int_0^{2\pi} \frac{f(\gamma_1(\theta_1), \gamma_2(\theta_2)) \gamma_1'(\theta_1) \gamma_2'(\theta_2)}{(\gamma_1(\theta_1) - z_1)(\gamma_2(\theta_2) - z_2)} d\theta_1 d\theta_2,$$

where the integrand is a continuous function of θ_1 and θ_2 for $(z_1, z_2) \in U_r(c)$. Hence by the formula of the iterated integral, this integral is equal to

$$\left(\frac{1}{2\pi i} \right)^2 \int_0^{2\pi} \frac{\gamma_1'(\theta_1)}{\gamma_1(\theta_1) - z_1} d\theta_1 \int_0^{2\pi} \frac{f(\gamma_1(\theta_1), \gamma_2(\theta_2)) \gamma_2'(\theta_2)}{\gamma_2(\theta_2) - z_2} d\theta_2.$$

Therefore by the Cauchy integral formula, the right-hand side of (1.10) becomes

$$\begin{aligned} & \left(\frac{1}{2\pi i} \right)^2 \int_{C_1} \frac{1}{\zeta_1 - z_1} d\zeta_1 \int_{C_2} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \\ &= \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1 = f(z_1, z_2), \end{aligned}$$

which proves (1.10) in this case. Similarly for general n , by a repeated application of the Cauchy integral formula, the right-hand side of (1.10) becomes

$$\begin{aligned} & \left(\frac{1}{2\pi i} \right)^n \int_{C_1} \frac{d\zeta_1}{\zeta_1 - z_1} \cdots \int_{C_{n-1}} \frac{d\zeta_{n-1}}{\zeta_{n-1} - z_{n-1}} \int_{C_n} \frac{f(\zeta_1, \dots, \zeta_{n-1}, \zeta_n)}{\zeta_n - z_n} d\zeta_n \\ &= \left(\frac{1}{2\pi i} \right)^{n-1} \int_{C_1} \frac{d\zeta_1}{\zeta_1 - z_1} \cdots \int_{C_{n-1}} \frac{f(\zeta_1, \dots, \zeta_{n-1}, z_n)}{\zeta_{n-1} - z_{n-1}} d\zeta_{n-1} \\ &= \cdots = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta_1, z_2, \dots, z_n)}{\zeta_1 - z_1} d\zeta_1 = f(z_1, \dots, z_n). \quad \blacksquare \end{aligned}$$

As in the case of holomorphic functions of one variable, we shall deduce the fundamental properties of holomorphic functions of n variables from the integral formula (1.10).

First let $\psi(\zeta_1, \dots, \zeta_n)$ be a continuous function on $C^n = C_1 \times \dots \times C_m$ and m_1, \dots, m_n natural numbers. Consider the integral

$$g(z) = \int_{C^n} \frac{\psi(\zeta_1, \dots, \zeta_n) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{m_1} \cdots (\zeta_n - z_n)^{m_n}} \quad (1.11)$$

as a function of $z = (z_1, \dots, z_n) \in U_r(c)$. Clearly $g(z)$ is continuous in $U_r(c)$. Then for fixed $z_2 \in U_{r_2}(c_1), \dots, z_n \in U_{r_n}(c_n)$, put

$$\varphi(\zeta_1) = \int_{C_2} \cdots \int_{C_n} \frac{\psi(\zeta_1, \dots, \zeta_n) d\zeta_2 \cdots d\zeta_n}{(\zeta_2 - z_2)^{m_2} \cdots (\zeta_n - z_n)^{m_n}}.$$

$\varphi(\zeta_1)$ is a continuous function of ζ_1 on C_1 . Hence

$$g(z) = g(z_1, z_2, \dots, z_n) = \int_{C_1} \frac{\varphi(\zeta_1)}{(\zeta_1 - z_1)^{m_1}} d\zeta_1 \quad (1.12)$$

is a holomorphic function of z_1 in $U_{r_1}(c_1)$. Similarly $g(z_1, \dots, z_n)$ is a holomorphic function of each variable z_k , $k = 1, \dots, n$, in $U_{r_k}(c_k)$. Hence $g(z) = g(z_1, \dots, z_n)$ is a holomorphic function of n variables z_1, \dots, z_n in the polydisk $U_r(c)$. By (1.12) we have

$$\begin{aligned} \frac{\partial}{\partial z_1} g(z_1, \dots, z_n) &= m_1 \int_{C_1} \frac{\varphi(\zeta_1)}{(\zeta_1 - z_1)^{m_1+1}} d\zeta_1 \\ &= m_1 \int_{C_1} \cdots \int_{C_n} \frac{\psi(\zeta_1, \dots, \zeta_n) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{m_1+1} \cdots (\zeta_n - z_n)^{m_n}}. \end{aligned} \quad (1.13)$$

Thus $(\partial g / \partial z_1)(z_1, \dots, z_n)$ is also holomorphic in $U_r(c)$. Similar results hold also for $\partial g / \partial z_k$.

By a repeated application of this result to the right-hand side of (1.10), we obtain the following theorem.

Theorem 1.2. A holomorphic function $f(z) = f(z_1, \dots, z_n)$ of n variables in a domain $D \subset C^n$ is arbitrarily many times differentiable in z_1, \dots, z_k in D , and all its partial derivatives $\partial^{m_1+\dots+m_n} f(z) / \partial z_1^{m_1} \cdots \partial z_n^{m_n}$ are holomorphic in D . Moreover taking a polydisk $U_r(c)$ such that $[U_r(c)] \subset D$, we have

$$\begin{aligned} \frac{\partial^{m_1+\dots+m_n}}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} f(z_1, \dots, z_n) \\ = \frac{m_1! \cdots m_n!}{(2\pi i)^n} \int_{C^n} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{m_1+1} \cdots (\zeta_n - z_n)^{m_n+1}} \end{aligned} \quad (1.14)$$

in $U_r(c)$. ■

As in the case of functions of one variable we denote by $f^{(m_1 \dots m_n)}(z_1, \dots, z_n)$ the partial derivative

$$\frac{\partial^{m_1 + \dots + m_n}}{\partial z_1^{m_1} \dots \partial z_n^{m_n}} f(z_1, \dots, z_n) \quad \text{of} \quad f(z) = f(z_1, \dots, z_n).$$

Theorem 1.3. *Let $f(z) = f(z_1, \dots, z_n)$ be a holomorphic function in a domain $D \subset \mathbb{C}^n$, and $c = (c_1, \dots, c_n) \in D$. Then in a polydisk $U_{\rho(c)} \subset D$ with centre c , $f(z)$ has a power series expansion in $z_1 - c_1, \dots, z_n - c_n$*

$$f(z) = \sum_{m_1, \dots, m_n=0}^{\infty} a_{m_1 \dots m_n} (z_1 - c_1)^{m_1} \dots (z_n - c_n)^{m_n}, \quad (1.15)$$

which is absolutely convergent in $U_{\rho}(c)$. The coefficient $a_{m_1 \dots m_n}$ is given by

$$a_{m_1 \dots m_n} = \frac{1}{m_1! \dots m_n!} f^{(m_1 \dots m_n)}(c_1, \dots, c_n). \quad (1.16)$$

Proof. By replacing z_k by $z_k - c_k$, $k = 1, \dots, n$, we may assume that $c_1 = c_2 = \dots = c_n = 0$. For a point $z = (z_1, \dots, z_n) \in U_{\rho}(0)$, $\rho = (\rho_1, \dots, \rho_n)$, take $r = (r_1, \dots, r_n)$ such that $|z_k| < r_k < \rho_k$ for $k = 1, \dots, n$. Then $z \in U_r(0)$, and $[U_r(0)] \subset U_{\rho}(0) \subset D$. Hence by (1.10)

$$f(z) = \left(\frac{1}{2\pi i} \right)^n \int_{C^n} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_1 \dots d\zeta_n}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)}. \quad (1.17)$$

Since $|\zeta_k| = r_k > |z_k|$, $k = 1, \dots, n$, we have

$$\frac{1}{\zeta_k - z_k} = \frac{1}{\zeta_k} \sum_{m_k=0}^{\infty} \left(\frac{z_k}{\zeta_k} \right)^{m_k}, \quad \left| \frac{z_k}{\zeta_k} \right| = \frac{|z_k|}{r_k} < 1, \quad k = 1, \dots, n.$$

Substituting these into the right-hand side of (1.17), we obtain a power series expansion

$$f(z) = \sum_{m_1, \dots, m_n=0}^{\infty} a_{m_1 \dots m_n} z_1^{m_1} \dots z_n^{m_n},$$

$$a_{m_1 \dots m_n} = \left(\frac{1}{2\pi i} \right)^n \int_{C^n} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_1 \dots d\zeta_n}{\zeta_1^{m_1+1} \dots \zeta_n^{m_n+1}}.$$

Letting M be the maximum of $|f(\zeta_1, \dots, \zeta_n)|$ on C^n , we have

$$|a_{m_1 \dots m_n}| \leq \frac{M}{r_1^{m_1} \dots r_n^{m_n}},$$

which proves that the above power series is absolutely convergent in $U_r(0)$. From (1.14), it is clear that $a_{m_1, \dots, m_n} = f^{(m_1, \dots, m_n)}(0) / m_1! \cdots m_n!$. ■

(b) Power Series

In this section we consider a power series

$$P(z) = P(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n=0}^{\infty} a_{m_1, \dots, m_n} z_1^{m_1} \cdots z_n^{m_n}$$

with centre 0. If $P(z)$ is convergent at z , we denote its sum by the same notation $P(z)$.

Theorem 1.4. *Let $w = (w_1, \dots, w_n)$ be such that $w_1 \neq 0, \dots, w_n \neq 0$. If $P(z)$ is convergent at $z = w$, then $P(z)$ is absolutely convergent for $|z_1| < |w_1|, \dots, |z_n| < |w_n|$, and its sum $P(z)$ is a holomorphic function of n variables z_1, \dots, z_n in $U_\rho(0)$ where $\rho = (|w_1|, \dots, |w_n|)$.*

Proof. For simplicity we consider the case $n = 2$. The general case is proved similarly. Since, by hypothesis, $P(z)$ is convergent, there exists a constant M such that $|a_{m_1, m_2} w_1^{m_1} w_2^{m_2}| \leq M < +\infty$. Hence

$$|a_{m_1, m_2}| \leq \frac{M}{\rho_1^{m_1} \rho_2^{m_2}},$$

where $\rho_1 = |w_1|$ and $\rho_2 = |w_2|$. Therefore if $|z_1| < \rho_1$ and $|z_2| < \rho_2$,

$$\sum_{m_1, m_2=0}^{\infty} |a_{m_1, m_2} z_1^{m_1} z_2^{m_2}| \leq M \sum_{m_1=0}^{\infty} \left(\frac{|z_1|}{\rho_1} \right)^{m_1} \sum_{m_2=0}^{\infty} \left(\frac{|z_2|}{\rho_2} \right)^{m_2} < +\infty,$$

namely, $P(z)$ is absolutely convergent. Moreover taking arbitrary r_1 and r_2 with $0 < r_1 < \rho_1$, $0 < r_2 < \rho_2$, we have

$$|a_{m_1, m_2} z_1^{m_1} z_2^{m_2}| \leq |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2}, \quad \sum_{m_1, m_2=0}^{\infty} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} < +\infty$$

for $z = (z_1, z_2)$ with $|z_1| < r_1$ and $|z_2| < r_2$. Therefore $P(z)$ is uniformly and absolutely convergent in $[U_r(0)]$ with $r = (r_1, r_2)$, hence continuous in $[U_r(0)]$. Since r_1 and r_2 are arbitrary real numbers with $0 < r_1 < \rho_1$, $0 < r_2 < \rho_2$, and $P(z)$ is clearly a holomorphic function in z_1 and z_2 separately, $P(z_1, z_2)$ is holomorphic in $U_\rho(0)$. ■

Replacing the variables z_k by $z_k - c_k$, $k = 1, \dots, n$, we obtain a power series

$$P(z - c) = P(z_1 - c_1, \dots, z_n - c_n) = \sum_{m_1, \dots, m_n=0}^{\infty} a_{m_1, \dots, m_n} (z_1 - c_1)^{m_1} \cdots (z_n - c_n)^{m_n}$$

with centre $c = (c_1, \dots, c_n)$.

Corollary. *If a power series $P(z - c)$ is convergent at $w = (w_1, \dots, w_n)$ with $w_1 \neq c_1, \dots, w_n \neq c_n$, $P(z - c)$ is absolutely convergent if $|z_k - c_k| < |w_k - c_k|$, $k = 1, \dots, n$, and its sum $P(z - c)$ is a holomorphic function in $U_\rho(c)$, where $\rho = (|w_1 - c_1|, \dots, |w_n - c_n|)$. ■*

The region of convergence of a power series $P(z - c)$ is the union $D = \bigcup U_\rho(c)$ of all polydisks $U_\rho(c)$ where $P(z - c)$ is absolutely convergent. A region of convergence D is a domain if it is not empty. In case $n = 1$, the region of convergence of a power series is an empty set, an open disk, or the whole \mathbb{C} itself, but in case $n \geq 2$, the region of convergence of a power series may take various forms.

The next theorem follows immediately from this Corollary and Theorem 1.3.

Theorem 1.5. *A function $f(z) = f(z_1, \dots, z_n)$ of n complex variables is holomorphic in a domain $D \subset \mathbb{C}^n$ if and only if for every point $c \in D$, $f(z)$ has a power series expansion $P(z - c)$ which is convergent in some neighbourhood of c . ■*

(c) Cauchy-Riemann Equation

First consider a continuously differentiable function $f(z)$ of one complex variable z in a domain $D \subset \mathbb{C}$. Decompose z and $f(z)$ into their real and imaginary parts by writing $z = x + iy$ and $f(z) = u + iv$. Then u and v are continuously differentiable functions of the real coordinates x, y in D . Using z and \bar{z} , we have

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}).$$

Here z and \bar{z} are *not* independent variables, but considering them as if they are independent, we define the partial derivatives of $f(z)$ with respect to z and \bar{z} by

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right). \quad (1.18)$$