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A. N. Parshin　I. R. Shafarevich (Eds.)

Algebraic Geometry V

Fano Varieties

代数几何 V

Fano 簇

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

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Fano Varieties

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Translated from the Russian
by Yu. G. Prokhorov and S. Tregub

Contents

Introduction

This survey continues the series of surveys devoted to the classification of algebraic varieties (Shokurov (1988), Shokurov (1989), Danilov (1988) Danilov (1989), Iskovskikh-Shafarevich (1989), Kulikov-Kurchanov (1989)). It deals with Fano varieties of dimension three and higher. The general classification problem was stated and partly advanced in the classical research of Italian geometers. In the last two decades the classification theory developed rapidly thanks to the new Mori theory of minimal models. It is based on the remarkable ideas and results due to S. Mori on extremal properties of cones of effective one-dimensional cycles (Mori (1982), Mori (1988)), using which the concept of a minimal model playing the central role in the classical birational classification of surfaces (see Iskovskikh-Shafarevich (1989)) was extended to varieties of higher dimension. Within the framework of the theory arises the category of projective varieties with some admissible singularities: terminal canonical, log canonical and others.

A minimal model in the sense of Mori is defined to be a normal projective variety with a numerically effective canonical divisor. According to the Mori Minimal Model Program, which is completely carried out in dimensions ≤ 3 and partly in dimensions ≥ 4 (see Mori (1982), Reid (1983a), Kawamata-Matsuda-Matsuki (1987), Clemens-Kollár-Mori (1988), Kollár et al. (1992), Wilson (1987a)), every irreducible algebraic variety over an algebraically closed field of characteristic zero is birationally equivalent either to a minimal model (if its Kodaira dimension ≥ 0) or to a fibration over a variety of smaller dimension (in particular, over a point) with rational singularities with the general fiber being a Fano variety (in this case the Kodaira dimension of the initial variety equals $-\infty$).

Therefore the Mori program establishes the important role that Fano varieties play in the birational classification of algebraic varieties. They are defined to be varieties with ample anticanonical class and form a subclass of varieties of Kodaira dimension $-\infty$.

The only one-dimensional Fano varieties are the projective line over an algebraically closed field and a conic over an arbitrary field. Two-dimensional Fano varieties are del Pezzo surfaces (see the survey of Iskovskikh-Shafarevich (1989)).

In connection with the problems of rationality and unirationality, G. Fano studied at the beginning of the century the class of varieties with canonical curve-sections (see Fano (1908), Fano (1915), Fano (1930), Fano (1931), Fano (1936), Fano (1942), Fano (1947)). Contemporary authors continued this study, taking for the definition of the class of varieties the ampleness of the anticanonical sheaf. G. Fano did not restrict himself to considering only nonsingular varieties, but for the present only nonsingular three-dimensional Fano varieties are classified in contemporary works. Although the problems of rationality and unirationality still remain important (and very difficult), at present it is the general problems of the structure theory that are of prime interest, namely, the classification of $\mathbb{Q}$-Fano varieties with admissible singularities in dimension three and higher, the problem of boundedness of the degree, solved for nonsingular Fano varieties of any dimension in Kollár-Miyaoka-Mori (1992c), Nadel (1991), Campana (1991a), the problems of uniruledness and rational connectedness, also solved for nonsingular Fano varieties of any dimension (see Miyaoka-Mori (1986), Kollár-Miyaoka-Mori (1992c), Campana (1992)), the study of Fano varieties with additional structures ($\mathbb{P}^r$-bundles, toric varieties, and others; see Batyrev (1981), Demin (1980), Szurek-Wiśniewski (1990c), Wiśniewski (1989b), Wiśniewski (1993)).

There are several rather complete expositions of the classification theory for nonsingular three-dimensional Fano varieties (see, for example, Iskovskikh (1979a), Iskovskikh (1988), Murre (1982), Mori-Mukai (1986), Mori-Mukai (1983a), Mukai (1992a)). Singular Fano varieties and Fano varieties of higher dimension have been studied in the last decade.

The goal of the present survey is to encompass as far as possible these separate results and to highlight the main directions and methods of research. We do not include in this survey the well-known (actually classical) results on two-dimensional Fano varieties, that is, nonsingular del Pezzo surfaces (see, for example, Nagata (1960), Manin (1972)), and on del Pezzo surfaces with canonical (see Du Val (1934), Demazure (1980), Hidaka-Watanabe (1981), Brenton (1980)) and log terminal singularities (see Alexeev (1988), Alexeev-Nikulin (1989), Nikulin (1989a), Nikulin (1988), Nikulin (1989), Alexeev (1994b)). We do not touch arithmetic results for Fano varieties (see Manin-Tsfasman (1986), Batyrev-Manin (1990), Manin (1993)), and the few known results in characteristic $p > 0$ (see Ballico (1989), Serpico (1980), Shepherd-Barron (1997)). The ground field k is assumed to be algebraically closed and of characteristic zero.

The Russian version of this survey was finished in 1995 and unfortunately many works on this subject appearing later were not included in it.

The survey begins with a brief exposition of some points of the Mori theory of minimal models of algebraic varieties, which assumes a central place in contemporary algebraic-geometric research. This is the contents of Chapter 1.

In Chapter 2 we give the basic definitions and examples, and formulate the simplest properties of Fano varieties which can be immediately deduced from the definition and general theorems such as the Riemann–Roch theorem, vanishing theorems etc. We also include in this chapter some general results on equations defining varieties connected with Fano varieties (canonical curves, varieties of minimal degree, intersections of quadrics). At the end of the chapter we reproduce some results on the existence of good divisors in anticanonical linear systems and on their base locus.

Chapter 3 is devoted to the description of the results due to T. Fujita on the classification of polarized del Pezzo varieties connected with n-dimensional Fano varieties of index $n-1$. We give also the proof of the main classification theorem.

In Chapter 4 we present the classification of three-dimensional Fano varieties with Picard number $\rho = 1$. The classification is based on the generalization of the classical method due to G. Fano of a double projection from a line, a conic, etc. with the use of Mori theory.

In Chapter 5 we describe the approach of N. P. Gushel and S. Mukai to the classification of Fano varieties with $\rho = 1$ using vector bundles. The method of vector bundles makes it possible to obtain a new proof of the classification theorem for three-dimensional Fano varieties with $\rho = 1$. This was done in Mukai (1988), Mukai (1989), Mukai (1992a), and in Gushel (1982), Gushel (1983), Gushel (1992) (only for genus $g = 6$ and 8).

Results on the uniruledness, the rational connectedness and the boundedness of the degree for n-dimensional Fano varieties are presented in chapter 6.

Chapter 7 is devoted to the classification of Fano varieties with $\rho \geq 2$. In the first section we describe the Mori–Mukai classification of three-dimensional Fano varieties with $\rho \geq 2$. In the second section we present some results related to the classification of Fano varieties of higher dimension with $\rho \geq 2$.

The problems of rationality for Fano varieties are discussed in chapters 8–10. We discuss briefly the basic methods for proving the non-rationality. In Chapter 8 we consider the method of intermediate Jacobians due to C. Clemens and Ph. Griffiths and the method connected with the Brauer group due to D. Mumford and M. Artin. In Chapter 9 we consider the method of factorization of birational maps (the classical Noether–Fano method and its generalization in the context of Mori theory). In Chapter 10 we collect the known general constructions of unirationality and rationality for Fano varieties and some concrete results as well.

In Chapter 11 we note some generalizations of Fano varieties known to us, describe some separate results not included in the main text, and give a list of open questions and problems.

The classification tables for del Pezzo varieties and nonsingular three-dimensional Fano varieties are placed in Chapter 12.

The first author worked on the final version of the survey during his visit to the Universities of Pisa and Genova. He would like to express his deep gratitude to the Departments of Mathematics and especially to Professors F. Bardelli, I. Bauer, F. Catanese and M. Beltrametti for their hospitality and the opportunity to work in excellent conditions. He also thanks the Italian Consiglio Nazionale delle Ricerche (CNR) for the financial support. The second author thanks the fund "Pro Mathematica" for the financial support. The present work was also partly financed by Grant No M30000 from the International Science Foundation. and by the Russian foundation for fundamental researches (project 93-011-1539).

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Chapter 1
Preliminaries

§1.1. Singularities

Let X be an irreducible normal projective variety of dimension $\dim X = d$ over an algebraically closed field k of characteristic zero. In this survey we shall use the following notation (see Kawamata-Matsuda-Matsuki (1987), Clemens-Kollár-Mori (1988)).

$Z_r(X)$ is the *group of r-dimensional cycles* on X, that is, the free abelian group generated by closed irreducible subvarieties of dimension $r, 0 \le r \le d-1$. In particular, $Z_{d-1}(X)$ is the *group of Weil divisors* on X.

$\mathrm{Div}(X)$ is the *group of Cartier divisors* on X. There is a natural injection (since X is normal)

$$\mathrm{Div}(X) \hookrightarrow Z_{d-1}(X) ,$$

the image consists of those divisors which locally in a neighborhood of every point can be determined by one equation. Elements of the group $Z_{d-1}(X) \otimes \mathbb{Q}$ are called *$\mathbb{Q}$-divisors*, and elements of the group $\mathrm{Div}(X) \otimes \mathbb{Q}$ are called *$\mathbb{Q}$-Cartier divisors*.

$\mathrm{Pic}(X)$ denotes, as usual, the *Picard group*, that is, the group of classes of Cartier divisors with respect to linear equivalence. This group is naturally isomorphic to the group of invertible sheaves (or, equivalently, line bundles) on X up to isomorphism.

A $\mathbb{Q}$-Cartier divisor $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ is said to be *big* if $h^0(X, \mathcal{O}_X(mD)) >$ const $\cdot\, m^d$ for (sufficiently large) $m \gg 0$ such that $mD \in \mathrm{Div}(X)$.

A $\mathbb{Q}$-Cartier divisor $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ is called *nef* (*numerically effective*) if $D \cdot C \ge 0$ for every complete curve $C \subset X$. The *intersection number* is defined as a rational number in the following way: let $m \in \mathbb{Z}$ be an integer such that $mD \in \mathrm{Div}(X)$; then

$$D \cdot C := \frac{1}{m} \deg(\mathcal{O}_X(mD)|_C) .$$

For any invertible sheaf $L \in \mathrm{Pic}(X)$, the degree $\deg(L|_C)$ is defined to be equal to $\deg \nu^* L$, where $\nu \colon \widetilde{C} \to C$ is the normalization of the curve C.

Cycles $z, z' \in Z_1(X)$ are said to be *numerically equivalent*, which is written as $z \equiv z'$, if $L \cdot z = L \cdot z'$ for every $L \in \mathrm{Pic}(X)$. By duality, one can define numerical equivalence in $\mathrm{Pic}(X)$. The pairing $\mathrm{Pic}(X) \times Z_1(X) \to \mathbb{Z}$ induces a perfect pairing

$$N^1(X) \times N_1(X) \to \mathbb{R} , \tag{1.1.1}$$

where $N^1(X) := \mathrm{Pic}(X)/(\mathrm{mod} \equiv) \otimes \mathbb{R}$, and $N_1(X) := Z_1(X)/(mod \equiv) \otimes \mathbb{R}$. For $\mathrm{Pic}(X)/(\mathrm{mod} \equiv) \otimes \mathbb{Q}$, we use the notation $N^1_{\mathbb{Q}}(X)$.

Let ω be a rational differential form of degree d on X. Then one can define the *Cartier divisor of this form* $(\omega)|_U$ on the open dense smooth subset $U = X - \operatorname{Sing} X$ ($\operatorname{codim} X \geq 2$ because X is normal). It can be extended to a Weil divisor on the whole X; the class of this Weil divisor is called *canonical* and is denoted by K_X or simply by K.

A variety X is called *Gorenstein* (or its singularities are called *Gorenstein*) if it is Cohen-Macaulay, and the dualizing sheaf ω_X is invertible. In such a case $K_X \in \operatorname{Pic}(X)$, that is, K_X is represented by a Cartier divisor (the converse in general is not true, see Ishii (1987), Ishii (1991)). If $mK_X \in \operatorname{Pic}(X)$ for some integer m, then X is called $\mathbb{Q}$*-Gorenstein.* The minimal positive integer m with this property is called the *(Gorenstein) index* of K_X (or X). All smooth varieties are, of course, Gorenstein.

Example 1.1.1. Let $F_4 \subset \mathbb{P}^5$ be the Veronese surface, that is, the image of $\mathbb{P}^2$ in $\mathbb{P}^5$ under the map determined by the complete linear system of conics $\mathcal{O}_{\mathbb{P}^2}(2)$, and let $X \subset \mathbb{P}^6$ be a cone over F_4. Then $\operatorname{Pic}(X) = \mathbb{Z} \cdot H$, where H is a hyperplane section. Denote by $E \subset X$ the Weil divisor which is the cone over the image of a line $l \in \mathbb{P}^2$ in F_4. Then $-K_X = 3E \notin \operatorname{Pic}(X)$, so the singularity in the vertex is not Gorenstein. But $2K_X = -6E = -2H \in \operatorname{Pic}(X)$, which means that X is $\mathbb{Q}$-Gorenstein of index 2. Note that locally near the vertex, X can be represented as $\mathbb{C}^3/r$, where r is the involution $(x, y, z) \to (-x, -y, -z)$. The expression $(dx \wedge dy \wedge dz)^{\otimes 2} = (dx^2 \wedge dy^2 \wedge dz^2)^{\otimes 2}/64x^2y^2z^2$ defines a 2-canonical form on X. If $f: \overline{X} \to X$ is the blow-up of the vertex $P \in X$, and if $F = f^{-1}(P) \simeq \mathbb{P}^2$ is the exceptional divisor, then $\overline{X}$ is nonsingular, and $K_{\overline{X}} = f^*K_X + \frac{1}{2}F$.

Definition 1.1.2. *A normal variety X is said to have at most canonical (respectively, terminal, log terminal, log canonical) singularities if it is $\mathbb{Q}$-Gorenstein, and for every resolution of singularities $f: X' \to X$ with exceptional divisors $E_i \subset X'$, the following conditions hold:*

$$mK_{X'} = f^*(mK_X) + \sum a_i E_i \tag{1.1.2}$$

(m is the index of X) with $a_i \geq 0$ (respectively, $a_i > 0, \frac{a_i}{m} > -1, \frac{a_i}{m} \geq -1$). Usually this formula is divided by m and is written as

$$K_{X'} = f^*(K_X) + \sum \alpha_i E_i, \quad \alpha_i = \frac{a_i}{m} \in \mathbb{Q}\,,$$

where $\alpha_i \geq 0$ for canonical (respectively, $\alpha_i > 0$ for terminal, $\alpha_i > -1$ for log terminal, and $\alpha_i \geq -1$ for log canonical) singularities. The numbers α_i are called discrepancies at E_i; they depend only on X and the (proper images of) divisors E_i, that is, they do not depend on the choice of resolution.

Let $\omega_X = \mathcal{O}(K_X)$ be the canonical (dualizing) sheaf. For any positive integer i, we denote by $\omega_X^{[i]}$ the double dual sheaf of the sheaf $\omega_X^{\otimes i}$ (it is taken to kill torsion and co-torsion, see Reid (1980b)). Then $\omega_X^{[i]}$ is a torsion-free sheaf of rank 1: it is locally free if and only if $i = am$ for some integer $a > 0$, where m is the index of K_X. If X has at most canonical singularities, and $f\colon X' \to X$ is a resolution, then $f_*(\omega_{X'}^{\otimes i}) = \omega_X^{[i]}$ for $i \geq 0$.

Proposition 1.1.3 (see, for example, Clemens-Kollár-Mori (1988)).

(i) *In dimension two, terminal points are nonsingular.*
(ii) *Two-dimensional canonical singularities are exactly the Du Val ones (they are also called rational double points). Locally in the complex topology they can be determined by one of the following equations:*

$$\begin{aligned} A_n:&\quad xy + z^{n+1} = 0, \quad n \geq 1\,;\\ D_n:&\quad x^2 + y^2 z + z^{n-1} = 0, \quad n \geq 4\,;\\ E_6:&\quad x^2 + y^3 + z^4 = 0\,;\\ E_7:&\quad x^2 + y^3 + yz^3 = 0\,;\\ E_8:&\quad x^2 + y^3 + z^5 = 0\,. \end{aligned} \tag{1.1.3}$$

Minimal resolutions of these singularities are described by the corresponding Dynkin diagrams; irreducible exceptional curves are represented by vertices, and two vertices are connected by an edge if the corresponding curves intersect. The number of vertices of the diagram is equal to n.

There is a complete list of terminal singularities in dimension 3 (refer to Mori (1985), Ried (1987), Kollár (1991)):

Theorem 1.1.4.

(i) *Three-dimensional terminal singularities are isolated points.*
(ii) *A three-dimensional hypersurface (i. e. Gorenstein) singularity is terminal if and only if it is isolated and is defined by an equation of the form:*

$$g(x, y, z) + th(x, y, z, t) = 0\,,$$

where g is one of the equations (1.1.3). Such singularities are usually called (compound Du Val) cDV-points (see Reid (1980b), Ried (1987)).
(iii) *Every three-dimensional terminal singularity is a quotient of some hypersurface terminal singularity (which is called a canonical $(m\!:\!1)$-cover, where m is the index of K_X at the singular point) by some cyclic group. The typical situation is:*

$$(xy + f(z^m, t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_m(1, -1, a, 0), \quad (a, m) = 1\,,$$

where $\mathbb{C}^N/\mathbb{Z}_m(a_1, \dots, a_N)$ denotes the quotient of $\mathbb{C}^N$ by the cyclic group action $(x_1, \dots, x_N) \to (\zeta^{a_1} x_1, \dots, \zeta^{a_N} x_N)$, and ζ is a primitive root of unity of degree m. Exceptional cases can be written in the form:

$$(x^2 + f(y, z, t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_m(a, b, c, d)$$

for some $m \leq 4$. There is a complete list of all possible cases.

(iv) *Every three-dimensional terminal singularity can be deformed to a set of terminal cyclic quotient singularities of the form $\mathbb{C}^3/\mathbb{Z}_m(1,-1,a)$, $(a,m)=1$.*

Remarks 1.1.5. (i) Two-dimensional log terminal singularities were studied in Brieskorn (1968). Iliev (1986), Kollár et al. (1992). They are exactly quotient singularities of $(\mathbb{C}^2, 0)$ by finite group actions. For log canonical singularities, see Kawamata (1988), Kollár et al. (1992).

(ii) The singularity in the vertex of a cone over the Veronese surface which was considered in Example 1.1.1 is terminal of index 2: it is isomorphic to the quotient singularity $\mathbb{C}^3/\mathbb{Z}_4(1,1,1)$ (see, for example, Wilson (1987a)).

(iii) All log terminal (in particular, terminal or canonical) singularities of any dimension are rational, that is, for some (every) resolution $f: X' \to X$, the equalities $R^i f_* \mathcal{O}_{X'} = 0$, $i > 0$, are true (see Elkik (1981), Kawamata-Matsuda-Matsuki (1987)). Log canonical singularities are not necessarily rational even in dimension 2 (see Kollár et al. (1992)).

(iv) Terminal singularities form the least possible class of singularities involving which the Mori Minimal Model Program is stated and can be true (as in dimension 3). Canonical singularities are exactly those which arise on canonical models.

(v) In the general Minimal Model Program (Kawamata-Matsuda-Matsuki (1987), Kollár et al. (1992)), more general types of singularities are defined and used. Namely, let $D = \sum \alpha_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X (D_i are irreducible Weil divisors) such that $K_X + D$ is a $\mathbb{Q}$-Cartier divisor. Then for every resolution $f: X' \to X$ we have:

$$K_{X'} \equiv f^*(K_X + D) + \sum \alpha_j E_j, \quad \alpha_j \in \mathbb{Q}\,, \tag{1.1.4}$$

where E_j are not necessarily only exceptional divisors.

A pair (X, D) or $K_X + D$ is called:

terminal if $\alpha_i > 0$;
canonical if $\alpha_i \geq 0$;
purely log terminal if $\alpha_i > -1$;
log canonical if $\alpha_i \geq -1$.

A pair (X, D) is called *Kawamata log terminal* if (X, D) is purely log terminal, and $\alpha_i < 1$ for all i.

The $\mathbb{Q}$-divisor $D = \sum \alpha_i {D'}_i$ is called the *boundary* if $0 \leq \alpha_i \leq 1$ $\forall i$.

The pair (X, D) is called a *log variety*, and $K_X + D$ is called a *log canonical divisor*.

In terms of discrepancies of only exceptional divisors, formula (1.1.4) can be rewritten in the form

$$K_{X'} + (f^{-1})_*(D) \equiv f^*(K_X + D) + \sum \alpha_i E_i, \quad \alpha_i \in \mathbb{Q}\,,$$

where the star in the subscript denotes the proper image of D as a Weil divisor. A resolution $f: X' \to X$ is called a *log resolution* of the log variety (X, D) if irreducible components of the support $\mathrm{Supp}((f^{-1})_*(D))$ are nonsingular and intersect transversally. Refer to Shokurov (1992), Kollár et al. (1992) and Kollár (1997) for different variants of the definition of log terminal objects in this more general situation.

§1.2. On Numerical Geometry of Cycles

Let X be a normal projective variety of dimension d. A cycle $z = \sum n_i z_i \in Z_i(X)$ is called *effective* if $n_i \geq 0\ \forall i$. Recall that we denote the numerical equivalence of 1-cycles with respect to intersections with Cartier divisors (and, by duality, the numerical equivalence of $\mathbb{Q}$-Cartier divisors) by the symbol $\equiv$.

A variety X is called $\mathbb{Q}$*-factorial* if some integral multiple of every Weil divisor is a Cartier divisor, that is, if $Z_{d-1}(X) \otimes \mathbb{Q} = \mathrm{Div}(X) \otimes \mathbb{Q}$.

The following notation is standard (see Clemens-Kollár-Mori (1988), Kawamata-Matsuda-Matsuki (1987)):

$N(X) := Z_1(X)/(mod \equiv) \otimes \mathbb{R}$;

$NE(X)$ is the least convex cone in $N(X)$ containing all effective 1-cycles;

$\overline{NE}$ is the closure of $NE(X)$ in the real topology; this is the so-called *Mori cone* of X;

$NS(X)$ is the *Néron-Severi group* of classes of Cartier divisors with respect to algebraic equivalence;

$\rho(X) := \mathrm{rk}(NS(X)) = \dim_{\mathbb{R}} N(X)$ is the Picard number.

Note that a numerically effective $\mathbb{Q}$-Cartier divisor D is big if and only if $D^d > 0$ (the self-intersection index of a $\mathbb{Q}$-Cartier divisor is defined as a rational number).

Assume now that X is $\mathbb{Q}$-Gorenstein. A half-line $R = \mathbb{R}_+[z] \subset \overline{NE}(X)$, $z \in Z_1(X) \otimes \mathbb{R}$ is called an *extremal ray* if:

(i) $-K_X \cdot z > 0$, and

(ii) from $z_1 + z_2 \in R$, $z_1, z_2 \in \overline{NE}(X)$ it follows that $z_1 \in R$ and $z_2 \in R$; this means that the ray R lies on the boundary of cone $\overline{NE}(X)$.

A rational curve $C \subset X$ is called an *extremal* curve if $\mathbb{R}_+[C]$ is an extremal ray and $0 < -K_X \cdot C \leq d+1$.

The important invariant of an extremal ray is the number $\mu(R) = \inf\{-K_X \cdot C \mid C \subset X$ is a rational curve whose numerical class is $[C] \in R\}$. This number is called the *length* of the extremal ray R.

An extremal ray R is called *numerically effective* if $C \cdot D \geq 0$ for every effective irreducible $\mathbb{Q}$-Cartier divisor D and a curve C such that $[C] \in R$.

For every $\mathbb{Q}$-Cartier divisor D, we set

$$\overline{NE}_D(X) = \{z \in \overline{NE}(X) \mid D \cdot z \geq 0\}\ ;$$

in particular, we shall call $\overline{NE}_K(X)$ the *positive part* of the Mori cone, and $\overline{NE}_{-K}(X) = \{z \in \overline{NE}(X) \mid K_X \cdot z \leq 0\}$ the *negative part*. By definition, all extremal rays lie in the negative part of the Mori cone. The following important result was proved by Kleiman (1962).

Theorem 1.2.1 (Kleiman's criterion for ampleness). *A $\mathbb{Q}$-Cartier divisor D on a variety X is ample if and only if $D \cdot z > 0$ for every $z \in \overline{NE}(X) - \{0\}$.*

For the anticanonical divisor $-K_X$ on a three-dimensional variety X, the ampleness criterion takes on the following simpler form.

Theorem 1.2.2 (Matsuki (1987)). *Let X be a complete normal variety of dimension 3 with at most canonical singularities. Assume that $|-mK_X| \neq \emptyset$ for some integer $m > 0$ (that is, $\kappa(-K_X) \geq 0$, where $\kappa(D)$ denotes the Kodaira dimension of a divisor D, see the definition below). Then the divisor $-K_X$ is ample if and only if $-K_X \cdot C > 0$ for every irreducible curve $C \in Z_1(X)$.*

The important notion in the Minimal Model Program is the numerical dimension of a numerically effective $\mathbb{Q}$-Cartier divisor $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$:

$$\nu(D) := \max\{m \mid D^m \not\equiv 0\}.$$

If K_X is nef, then $\nu(K_X)$ is called the *numerical dimension* of the variety X and is often denoted by $\nu(X)$. The numerical dimension is closely related to the Kodaira dimension. Recall the definition of the latter.

For any Cartier divisor $D \in \mathrm{Pic}(X)$, denote by $\varphi_{|D|}\colon X \dashrightarrow \mathbb{P}^{\dim |D|}$, as usual, the rational map determined by the complete linear system $|D|$. The *Kodaira D-dimension* $\kappa(X, D)$ is defined as follows (it is also called the *Iitaka D-dimension*):

$$\kappa(X,D) := \begin{cases} \max\{\dim \varphi_{|mD|}(X)\} & \text{if } |mD| \neq 0 \text{ for some integer } m > 0; \\ -\infty & \text{otherwise.} \end{cases}$$

We remark that $\kappa(X, D)$ can be characterized by the property: there exist $\alpha, \beta > 0$ and $m_0 \in \mathbb{Z}$, $m_0 > 0$, such that the following inequalities hold for $m \gg 0$:

$$\alpha m^\kappa \leq h^0(X, \mathcal{O}_X(mm_0D)) \leq \beta m^\kappa \ .$$

Let $R(X, D) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$ be a graded algebra with respect to the tensor multiplication of sections; then

$$\kappa(X,D) = \begin{cases} (\text{transcendence degree of } R(X,D)) - 1 & \text{if } R(X,D) \neq k; \\ -\infty & \text{otherwise.} \end{cases}$$

A divisor $D \in \mathrm{Div}(X)$ is big if $\kappa(X, D) = d = \dim X$.

The *Kodaira dimension* $\kappa(X)$ of X is defined to be equal to $\kappa(X', K_{X'})$, where X' is any complete nonsingular variety birationally isomorphic to X