n-BODY PROBLEMS AND MODELS

Donald Greenspan

n-BODY
PROBLEMS AND MODELS

Donald Greenspan
University of Texas at Arlington, USA

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: Suite 202, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

N-BODY PROBLEMS AND MODELS

ISBN 981-238-722-6

Printed in Singapore by World Scientific Printers (S) Pte Ltd

n-BODY
PROBLEMS AND MODELS

Preface

This book is concerned with computer simulation of scientific and engineering phenomena in a fashion which is consistent with the two principles:

(1) All things change with time, and
(2) All material bodies consist of atoms and/or molecules.

Applications include solitons, crack development, biological sorting, saddle surfaces, rotating tops, bubbles in liquids, liquid surface adhesion, relativistic oscillation and development of turbulent flows. Theoretically we develop discrete equations with conservation laws which are identical to those of continuum mechanics. Our molecular studies are in complete accord with modern nanophysics.

Graduates and professional researchers in mathematics, physics, materials science, fluid dynamics, and electrical and mechanical engineering will find this book a contemporary resource for their work on modelling physical phenomena.

Finally, I wish to thank Ann Kostant, Executive Editor, Mathematics and Physics, Birkhauser, Boston, for permission to use material from my book PARTICLE MODELING (1997).

Donald Greenspan
Arlington, Texas 2004

Problem Statement

The general N-body problem is usually formulated for $N \geq 2$ as follows. In cgs units and for $i = 1, 2, \ldots, N$, let P_i of mass m_i be at $\vec{r}_i = (x_i, y_i, z_i)$, have velocity $\vec{v}_i = (v_{i,x}, v_{i,y}, v_{i,z})$, and have acceleration $\vec{a}_i = (a_{i,x}, a_{i,y}, a_{i,z})$ at time $t \geq 0$. Let the positive distance between P_i and P_j, $i \neq j$, be $r_{ij} = r_{ji} \neq 0$. Let the force on P_i due to P_j be $\vec{F}_{ij} = \vec{F}_{ij}(r_{ij})$, so that the force depends only on the *distance* between P_i and P_j. Also, assume that the force $\vec{F}_{ji}$ on P_j due to P_i satisfies $\vec{F}_{ji} = -\vec{F}_{ij}$. Then, given the initial positions and velocities of all the $P_i, i = 1, 2, 3, \ldots, N$, the general N-body problem is to determine the motion of the system if each P_i interacts with all or part of the other P_j's in the system.

The prototype N-body problem was formulated around 1900 and was a collisionless problem. In it the P_i were the sun and the then known eight planets and the force on each P_i was gravitational attraction. However, since 1900 and up to the present, a variety of other N-body problems have come to be of interest in the sciences and in engineering. These problems will be categorized according to the choices $N = 1, 2 \leq N \leq 100, 100 < N \leq 10000, 10000 < N$. These categories have been determined in accordance with the capabilities of a Digital Alpha 533 personal scientific computer, which has been used for all the examples to be described.

Note immediately that a 1-body problem is not a special case of the general N-body problem, which has been formulated only for $N \geq 2$.

Finally, observe that each of the models to be studied is nonlinear. Linear models often have only limited life spans which end when refinement becomes essential.

Contents

Chapter 1

The 1-Body Problem

1.1. Nonlinear Oscillation

An important class of 1-body problems is found in the study of nonlinear oscillators. An oscillator is a body which moves up and back over all or part of a finite path. The prototype nonlinear oscillator is the swinging pendulum and in this section we turn attention to it. Other nonlinear oscillators can be treated in the fashion to be developed.

Consider a pendulum, as shown in Figure 1.1, which has mass m centered at P and is hinged at O. Assume that P is constrained to move on a circle of radius l whose center is O. Let θ be the angular measure, in radians, of the pendulum's deviation from the vertical. The problem is that of describing the motion of P after release from a position of rest.

It is known from laboratory experiments that the motion of the pendulum is damped and that the length of time between successive swings decreases.

Using cgs units, we reason analytically as follows. Assume that the motion of P is determined by Newton's dynamical equation

$$F = ma. \tag{1.1}$$

Circular arc NP has length $l\theta$, so that $a = \frac{d^2}{dt^2}(l\theta) = l\ddot{\theta}$. Thus, (1.1) becomes

$$F = ml\ddot{\theta}. \tag{1.2}$$

In considering the force F which acts on P, let F_1 be the gravitational component, so that

$$F_1 = -mg\sin\theta, \quad g > 0, \tag{1.3}$$

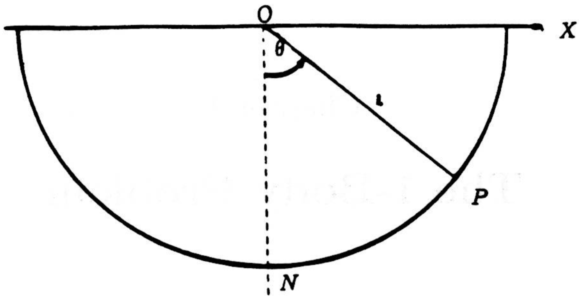

Figure 1.1. A pendulum.

and let F_2 be a damping component of the form

$$F_2 = -c\dot{\theta}, \ c \text{ a nonnegative constant.} \tag{1.4}$$

Assume that these are the only forces whose effects are significant. Then

$$F = -mg\sin\theta - c\dot{\theta},$$

so that (1.2) reduces readily to

$$\ddot{\theta} + \frac{c}{ml}\dot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{1.5}$$

The problem, then, is one of solving (1.5) subject to given initial conditions

$$\theta(0) = \alpha, \quad \dot{\theta}(0) = 0. \tag{1.6}$$

For illustrative purposes, let us consider the strongly damped pendulum motion defined by

$$\ddot{\theta} + (0.3)\dot{\theta} + \sin\theta = 0 \tag{1.7}$$

$$\theta(0) = \frac{1}{4}\pi, \quad \dot{\theta}(0) = 0. \tag{1.8}$$

No analytical method is known for constructing the exact solution of this problem. Numerically, then, set $t = x$ and $\theta = y$ and solve (1.7) with $\Delta t = 0.01$ using Kutta's fourth order formulas, which are given in generic form in Appendix I. The computation is carried out for 15000 time steps, that is, for 150 seconds of pendulum motion. The first 15.0 seconds of pendulum oscillation is shown in Figure 1.2, where the peak, or extreme, values 0.78540, −0.47647, 0.29335, −0.18156, 0.11259 occur at the times 0.00, 3.28, 6.49, 9.68, 12.86, respectively. The time required for the pendulum to travel

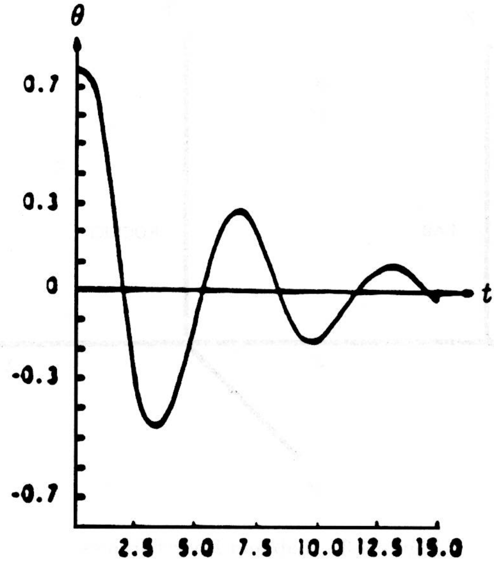

Figure 1.2. Damped pendulum motion.

from one peak to another decreases monotonically and damping is present during the entire simulation, in agreement with experimentation.

Any attempt to linearize (1.7) results in an analytical solution which either does not damp out, or has a constant time interval between successive swings, or both.

1.2. Concepts from Special Relativity

Interestingly enough, there exist some very important 1-body problems in Special Relativity. Let us then turn to a basic dynamical problem in Special Relativity and begin by discussing the very few concepts which will be required for the development. Incidentally, Special Relativity does not allow N-body problems for $N > 1$ because simultaneity is not a property of this branch of physics.

In Special Relativity one takes into account the time required for light to travel from a phenomenon being observed to the eye of the observer. Consider, then, two reference frames: a *lab* frame with Euclidean coordinates X, Y, Z and a *rocket* frame with Euclidean coordinates X', Y', Z', which coincide initially. In the frames one positions observers O and O' at

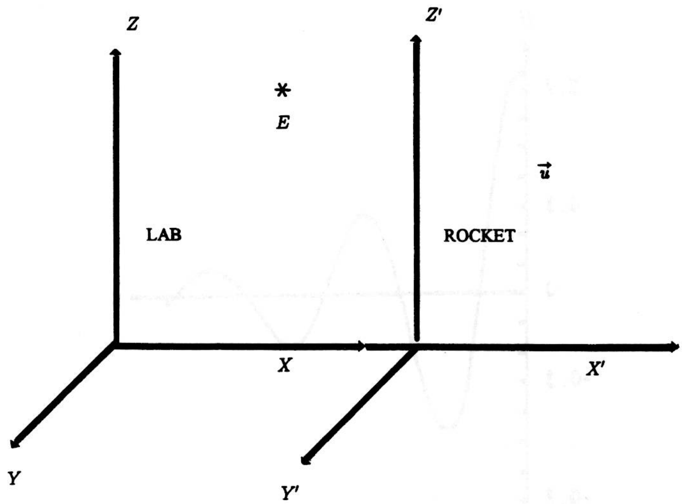

Figure 1.3. Lab and Rocket frames.

their respective origins. At some initial time the observers have synchronized clocks. Assume the rocket frame is in motion in the X direction with speed u relative to the lab frame. (See Figure 1.3) Assume that $|u|$ is less than the speed of light.

An event E, like an exploding star, is observed by both O and O'. O records E as happening at (x, y, z) at time t, while O' records E as happening at (x', y', z') at time t'. Taking into account the time for light to travel to the eyes of the observers, these variables are related by the Lorentz transformation (Bergmann (1976)):

$$\begin{aligned} x' &= \frac{c(x-ut)}{(c^2-u^2)^{1/2}}, \quad y' = y, \quad z' = z, \\ t' &= \frac{(c^2t-ux)}{c(c^2-u^2)^{1/2}}, \quad |u| < c, \end{aligned} \tag{1.9}$$

or, equivalently,

$$\begin{aligned} x &= \frac{c(x'+ut')}{(c^2-u^2)^{1/2}}, \quad y = y', \quad z = z', \\ t &= \frac{(c^2t'+ux')}{c(c^2-u^2)^{1/2}}, \quad |u| < c, \end{aligned} \tag{1.10}$$

in which c is the speed of light.

For *covariance* relative to the Lorentz transformation, Einstein showed that for the motion of a particle P of rest mass m_0,

$$F = \frac{d}{dt}(mv), \quad m = \frac{cm_0}{(c^2 - v^2)^{1/2}}, \quad |v| < c, \qquad \text{(lab)} \tag{1.11}$$

maps under the Lorentz transformation into

$$F' = \frac{d}{dt'}(m'v'), \quad m' = \frac{cm_0}{(c^2 - v'^2)^{1/2}}, \quad |v'| < c, \qquad \text{(rocket)} \tag{1.12}$$

that is, the laws of motion are the *same* in both the lab frame and the rocket frame. For future use and because of its basic importance, let us actually prove this result.

We first define the continuum concepts of velocity and acceleration. In the lab frame, set

$$v = \frac{dx}{dt}, \quad a = \frac{dv}{dt}, \quad |v| < c, \tag{1.13}$$

while in the rocket frame set

$$v' = \frac{dx'}{dt'}, \quad a' = \frac{dv'}{dt'}, \quad |v'| < c. \tag{1.14}$$

To relate v and v', we have from (1.9)

$$v' = \frac{dx'}{dt'} = \frac{c^2(dx - udt)}{c^2dt - udx} \tag{1.15}$$

so that

$$v' = \frac{c^2(v - u)}{c^2 - uv}. \tag{1.16}$$

Equivalently, from (1.10), one finds

$$v = \frac{c^2(v' + u)}{c^2 + uv'}. \tag{1.17}$$

Similarly, the relationship between a and a' is found to be

$$a' = \frac{c^3(c^2 - u^2)^{3/2}}{(c^2 - uv)^3}a, \tag{1.18}$$

or, equivalently,

$$a = \frac{c^3(c^2 - u^2)^{3/2}}{(c^2 + uv')^3}a'. \tag{1.19}$$

We are now ready to prove the Einstein result which is formulated in the following theorem.

Theorem 1.1. *Let a particle P be in motion along the X axis in the lab and along the X' axis in the rocket. In the lab frame let the mass m of P be given by*

$$m = \frac{cm_0}{(c^2 - v^2)^{1/2}}, \quad |v| < c, \tag{1.20}$$

where m_0 is a positive constant called the rest mass of P and v is the speed of P in the lab. In the rocket frame let the mass m' of P be given by

$$m' = \frac{cm_0}{(c^2 - (v')^2)^{1/2}}, \quad |v'| < c, \tag{1.21}$$

where m_0 is the same constant as in (1.20) and v' is the speed of P in the rocket. Let a force F be applied to P in the lab. In rocket coordinates denote the force by F', so that

$$F = F'.$$

Then, if in the lab the equation of motion is given by

$$F = \frac{d}{dt}(mv), \tag{1.22}$$

it follows that in the rocket the equation of motion of P is

$$F' = \frac{d}{dt'}(m'v'). \tag{1.23}$$

Proof. From (1.22) and (1.20)

$$\begin{aligned} F &= v\frac{dm}{dt} + m\frac{dv}{dt} \\ &= \frac{v^2\, ma}{c^2 - v^2} + ma \end{aligned}$$

so that

$$F = \left(\frac{c^2}{c^2 - v^2}\right) ma. \tag{1.24}$$

From (1.15) and (1.13), then, we must have

$$F' = \left(\frac{c^2}{c^2 - (v')^2}\right) m'a'. \tag{1.25}$$

Since $F = F'$, the proof will follow if

$$\left(\frac{c^2}{c^2 - v^2}\right) ma \equiv \left(\frac{c^2}{c^2 - (v')^2}\right) m'a'. \tag{1.26}$$

However, substitution of (1.16), (1.18) and (1.21) into the right side of (1.26) yields, quite remarkably, that the identity is valid and the theorem is proved. □

1.3. Relativistic Oscillation

Let us consider now a particle P which oscillates on the X axis in the lab frame. Assume that the force F on P is one whose magnitude depends only on the x coordinate of P. Then, let

$$F = f(x). \tag{1.27}$$

Assume that initially, that is, at time $t = 0, P$ is at x_0 and has speed v_0. Then the equation of motion for P in the lab frame is

$$\frac{d}{dt}(mv) = f(x), \tag{1.28}$$

or,

$$\left(\frac{c^2}{c^2 - v^2}\right) ma = f(x), \tag{1.29}$$

or,

$$c^2 m\ddot{x} = f(x)(c^2 - \dot{x}^2). \tag{1.30}$$

In turn, the latter equation reduces to

$$c^3 m_0 \ddot{x} = f(x)(c^2 - \dot{x}^2)^{3/2},$$

so that, finally, we find

$$\ddot{x} - \frac{f(x)}{c^3 m_0}(c^2 - \dot{x}^2)^{3/2} = 0, \tag{1.31}$$

is the differential equation one has to solve in the lab frame, given the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0. \tag{1.32}$$

In general, (1.31) cannot be solved in closed form, so that the observer in the lab must now introduce a computer to approximate the solution. However, the observer in the rocket frame also observes the motion of P, but in his coordinate system. His equation and initial conditions are found by applying (1.10), (1.17) and (1.19) to (1.31) and (1.32). Thus, he too will be confronted with a problem which requires a computer and so a computer identical to that in the lab is now introduced into the rocket.

The fundamental problem which now arises is: How should the computations be done in the lab and rocket so that the physics of Special Relativity is preserved, that is, so that the *numerical* results will be related by the Lorentz transformation. We show now how this can be done.

1.4. Numerical Methodology

For $\Delta t > 0$, let $t_k = k\Delta t, k = 0, 1, 2, 3 \ldots$. Let t'_k correspond to t_k by the Lorentz transformation. At t_k let P be at (x_k, y_k, z_k) in the lab and at (x'_k, y'_k, z'_k) in the rocket. These are also related by the Lorentz transformation. Define

$$v_k = \frac{\Delta x_k}{\Delta t_k} = \frac{x_{k+1} - x_k}{t_{k+1} - t_k}, \quad a_k = \frac{\Delta v_k}{\Delta t_k} = \frac{v_{k+1} - v_k}{t_{k+1} - t_k}, \quad \text{(LAB)} \tag{1.33}$$

$$v'_k = \frac{\Delta x'_k}{\Delta t'_k} = \frac{x'_{k+1} - x'_k}{t'_{k+1} - t'_k}, \quad a'_k = \frac{\Delta v'_k}{\Delta t'_k} = \frac{v'_{k+1} - v'_k}{t'_{k+1} - t'_k} \quad \text{(ROCKET)} \tag{1.34}$$

Then corresponding to (1.16)–(1.19), one has by direct substitution that

$$v'_k = \frac{c^2(v_k - u)}{c^2 - uv_k}, \quad v_k = \frac{c^2(v'_k + u)}{c^2 + uv'_k} \tag{1.35}$$

$$\begin{aligned} a'_k &= \frac{c^3(c^2 - u^2)^{3/2}}{(c^2 - uv_k)^2(c^2 - uv_{k+1})} a_k, \\ a_k &= \frac{c^3(c^2 - u^2)^{3/2}}{(c^2 + uv'_k)^2(c^2 + uv'_{k+1})} a'_k. \end{aligned} \tag{1.36}$$

In the limit (1.33)–(1.36) converge to (1.13), (1.14) and (1.16)–(1.19). Our problem now is to choose an approximation for

$$F = \frac{d}{dt}(mv) \tag{1.37}$$

in the lab which will transform covariantly into the rocket. The clue for such a choice comes from (1.24) which is equivalent to (1.22). What we