Mathematics and Its Applications

K. Vajravelu (Ed.)

Differential Equations and Nonlinear Mechanics

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Edited by

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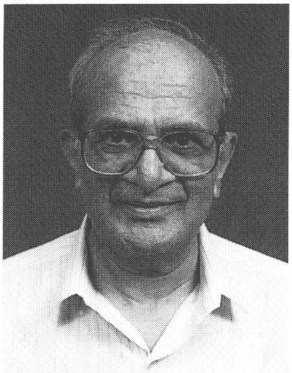

Professor V. Lakshmikantham has had an illustrious career as a mathematician, administrator, teacher, and a promoter of mathematics. As a mathematician, Professor Lakshmikantham has published hundreds of articles and dozens of books and monographs in the areas of differential equations, dynamical systems, and nonlinear analysis. His work has had an international impact on research in these areas. Currently many mathematicians and followers worldwide are expanding and developing his ideas.

As an administrator, he is well known and held in high regard for his leadership and motivational skills. He has served as department head in several universities. He has been able to construct, and convert obscure departments into prominent Ph.D. granting research departments, which have received national renown. Professor Lakshmikantham has achieved this with his extraordinary leadership and motivational skills, and without a large influx of funds. He has encouraged inactive faculty to start doing research and enjoy their productive careers.

As a teacher, Professor Lakshmikantham is a master in motivating and inspiring his students to perform at their highest capabilities. To him, each student is a member of his family with none treated better than another. He follows his students' careers and lives wherever they may go. As a result, most of his students have become productive citizens of the mathematical community and several of them are serving as chief editors of mathematical journals.

Professor Lakshmikantham has devoted his whole life to mathematics. He had envisioned the importance of unifying and solidifying the area of nonlinear analysis. The Journal of Nonlinear Analysis and other journals that he founded have been useful and timely.

Furthermore, Professor Lakshmikantham has touched many lives academically, intellectually, and professionally. He has played an important role in the development and nurturing of careers of mathematicians all over the world. His nurturing nature and fine people skills have had an effect on everyone who have been in his presence. Professor Lakshmikantham will live forever, in the heart, spirit, and minds of his family, friends, and students.

PREFACE

The International Conference on Differential Equations and Nonlinear Mechanics was hosted by the University of Central Florida in Orlando from March 17-19, 1999. One of the conference days was dedicated to Professor V. Lakshmikantham in honor of his 75th birthday. 50 well established professionals (in differential equations, nonlinear analysis, numerical analysis, and nonlinear mechanics) attended the conference from 13 countries. Twelve of the attendees delivered hour long invited talks and remaining thirty-eight presented invited forty-five minute talks. In each of these talks, the focus was on the recent developments in differential equations and nonlinear mechanics and their applications. This book consists of 29 papers based on the invited lectures, and I believe that it provides a good selection of advanced topics of current interest in differential equations and nonlinear mechanics.

I am indebted to the Department of Mathematics, College of Arts and Sciences, Department of Mechanical, Materials and Aerospace Engineering, and the Office of International Studies (of the University of Central Florida) for the financial support of the conference. Also, to the Mathematics Department of the University of Central Florida for providing secretarial and administrative assistance. I would like to thank the members of the local organizing committee, Jeanne Blank, Jackie Callahan, John Cannon, Holly Carley, Brad Pyle, Pete Rautenstrauch, and June Wingler for their assistance. Thanks are also due to the conference organizing committee, F.H. Busse, J.R. Cannon, V. Girault, R.H.J. Grimshaw, P.N. Kaloni, V. Lakshmikantham, R.N. Mohapatra, D. Nicholson, K.R. Rajagopal, and A. Sequeira. The invited speakers of the conference, especially Shair Ahmad who delivered the banquet talk, and everyone who attended the conference deserve a special mention for making this a success. My special thanks are due to Jackie Callahan for typing the manuscript carefully. Also, I wish to thank J.R. Martindale, Editor, and the staff of Kluwer Academic Publishers.

Finally, I thank my wife, Rani, for her ideas, devotion, and for having a vision for me; and my older son, Ravy, for his computer assistance, and to my younger son, Gopi, for his support and understanding, throughout the stages of the conference.

K. Vajravelu

Orlando, Florida
June 2000

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1. INTRODUCTION

The radius of stability and the radius of instability of the zero solution of the differential equation $x' = f(t,x)$ were introduced by Salvadori and Visentin [9], [10]. These radii in some sense provide a measure of the "region" of stability or instability of the zero solution. This knowledge has been used in the study of small solutions $x_p(t)$ of perturbations of the differential equation $x' = f(t,x)$ given by $x_p' = f(t,x_p) + h(t,x_p)$. In particular a relationship between the radius of stability of the zero solution of $x' = f(t,x)$ and its total stability was also introduced in [9] and [10]. Having been motivated by mechanical systems subject to conservative perturbations these authors analyzed the total stability of $x' = f(t,x)$ using the perturbed differential equation $x' = g(t,x,\lambda)$ where $g(t,x,0) = f(t,x)$ and λ is a parameter in some Banach Space β. In this paper we often will assume β is the real line.

In this paper we wish to study the continuity properties of the radii of stability and instability in terms of the total stability of the zero solution of $x' = f(t,x)$. We generally only consider the scalar case (although some extension to higher dimensions are discussed). We provide results in the case where the perturbed differential equation has a bifurcation phenomenon and study the properties of the radii of stability and instability in this case.

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2. PRELIMINARIES

Consider the unperturbed differential equation given by

$$x' = f(t,x) \tag{2.1}$$

where $f:[R\times D, R]$ and D is a neighborhood of the origin.

Let the perturbations of (2.1) be given by

$$x_p' = g(t,x_p,\lambda) \tag{2.2}$$

where $g(t,x,\lambda)$ is scalar in x and $g\in C[R\times D\times\wedge,\mathbb{R}]$ for some set $\Lambda\subseteq\mathbb{R}$ such that the origin is an accumulation point of $\wedge$. We also assume that

$$(*)\qquad \begin{cases} a(|\lambda|)\le \|g(t,x,\lambda)-f(t,x)\|\le b(|\lambda|), & \text{where} \\ b(\cdot),a(\cdot):\mathbb{R}^+\to\mathbb{R}^+ & \text{with} \\ a(0)=b(0)=0 \end{cases}$$

in which both $a(\cdot)$ and $b(\cdot)$ are strictly increasing and continuous.

We shall often restrict our attention to the case in which the unperturbed system is the autonomous differential equation

$$x' = f(x) \tag{2.3}$$

where $f\in C[D,\mathbb{R}]$. Assume also that $f(0)=0$.

Let the perturbations of (2.3) be given by

$$x_p' = g(x_p,\lambda), \tag{2.4}$$

where $g(x,\lambda)$ satisfies $(*)$.

Denote the solutions of (2.3) through (t_0,x_0) by $x(t,t_0,x_0)$ and denote the solutions of (2.4) through $(t_0,x_{p,0})$ by $x_p(t,t_0,x_{p,0})$.

The following definitions of the radius of stability and the radius of instability of the zero solution were given by Salvadori and Visentin [10]. (See also [11].) For completeness we shall consider (2.1) and provide the definition in the case in which $x\in\mathbb{R}^n$.

Definition 1. [11] The *radius of stability* is defined to be $r(t_0,\varepsilon)$ where

$$r(t_0,\varepsilon)=\sup\{\delta\ge 0:\|x_0\|<\delta : \text{implies that } \|x(t,t_0,x_0)\|<\varepsilon \text{ for all } t\ge t_0\}.$$

Definition 2. [11] The *radius of uniform stability* is defined to be $r(\varepsilon)$ where

$$r(\varepsilon)=\inf\{r(t_0,\varepsilon): t_0\in I\}.$$

Definition 3. [11] The *radius of instability* is defined to be $R(t_0)$ where

$R(t_0) = \{\sup \eta > 0$: there exists two sequences $\{x_i\}$ and $\{t_i\}$ *with* $x_i \in D$ and $t_i \in I$ for all $i \in \mathbb{R}$ such that $\|x(t_i, t_0, x_i)\| \leq \eta$ for all $i \in \mathbb{N}\}$.

Definition 4. [11] The *radius of non-uniform stability* is defined to be R where

$R = \{\sup \eta \geq 0$: there exists three sequences $\{x_i\}, \{t_i\}$ and $\{t_{0,i}\}$ such that $\|x(t_i, t_{0,i}, x_i)\| \geq \eta$ for all $i \in \mathbb{N}$, where $\lim_{i \to \infty} \|x_i\| = 0$ and $\lim_{i \to \infty} t_{0,i} = \infty$ and $t_i \geq t_{0,i}$ for all $i \in \mathbb{N}\}$.

The following definition of conditional total stability was given by Salvadori and Visentin, [9], [10].

Definition 5. The zero solution, $x \equiv 0$, of (2.1) is conditionally totally stable if for each $\varepsilon > 0$ *and* $t_0 \geq 0$ there exists $\delta_1(t_0, \varepsilon) \geq 0$ *and* $\delta_2(t_0, \varepsilon) \geq 0$ such that for each $|x_0| < \delta_1$ and for each $\lambda \in \Lambda$ *with* $|\lambda| < \delta_2$ the solution $x_p(t, t_0, x_0)$ of (2.2) satisfies

$$\|x_p(t, t_0, x_0)\| < \varepsilon \text{ for every } t \geq t_0.$$

Theorem 1. ([2]) The zero solution of (2.3) is uniformly totally stable if and only if there exists a nested family of contracting compact neighborhoods of the origin which are invariant and asymptotically stable.

A bifurcation theorem given in [6] is now presented.

Theorem 2. Suppose that $g(0, \lambda) \equiv 0$ and the origin of (2.4) is asymptotically stable for $\lambda = 0$ and completely unstable for $\lambda > 0$. Then there exists $\lambda^* > 0$ and a neighborhood T of the origin such that if $\lambda \in (0, \lambda^*)$ *and* M_λ is the largest invariant compact subset of $T \setminus \{0\}$, *then* (M_λ) is a family of asymptotically stable compact sets bifurcating from $\{0\}$.

3. PRIMARY RESULTS

We now present some properties of the radii of stability and instability. We use $|\cdot|$ to denote the norm in one dimension.

Proposition 1. Suppose that the zero solution of (2.1) is conditionally totally stable. Then for each $t_0 \in I$ and for each $\varepsilon > 0$ there exists $\delta(t_0, \varepsilon) > 0$ such that $r(t_0, \varepsilon, \lambda)$ need not be continuous in λ *for each* λ *with* $|>| \in (0, \delta)$.

An example is given which depicts Proposition 1.

Example 1. Let $x' = f(x)$ where

$$f(x) = \begin{cases} (-1)^n\left(-x + \dfrac{1}{n+1}\right) & x \in \left(\dfrac{1}{n+1}, \dfrac{2n+1}{2n(n+1)}\right) \\ (-1)^n\left(x - \dfrac{1}{n}\right) & x \in \left[\dfrac{2n+1}{2n(n+1)}, \dfrac{1}{n}\right) \\ 0 & x = \dfrac{1}{n} \end{cases} . \tag{3.1}$$

An application of Theorem 1 implies the zero solution is uniformly totally stable and hence, conditionally uniformly stable. Fix $t_0 \geq 0$. Note that for each $x_0 = \frac{1}{n}$ the solution $x(t, t_0, x_0)$ satisfies

$$x(t, t_0, x_0) \equiv \frac{1}{n}.$$

Now let $\varepsilon = \frac{1}{2k-1}$ for some $k \in \mathbb{N}$. Fix $t_0 \geq 0$. Now consider the perturbation of (3.1) given by

$$x_p' = f(x_p) + \lambda \tag{3.2}$$

where $\lambda \in \left(0, \frac{1}{4k}\right)$. And so for each $\lambda \in \left(0, \frac{1}{4k}\right)$

$$r(t_0, \varepsilon, \lambda) \leq \frac{1}{2k}.$$

This is due to the fact that the right hand side of (3.2) is positive for all $x \in \left(\frac{1}{2k}, \frac{1}{2k-1}\right]$ and hence, implies that for any x_0 with

$$x_0 \in \left(\frac{1}{2k}, \frac{1}{2k-1}\right]$$

the solution $x(t)$ satisfies for some $T(x_0) > 0$

$$x(T, t_0, x_0) \geq \varepsilon.$$

And so

$$\lim_{\lambda \to 0^+} r(t_0, \varepsilon, \lambda) = \frac{1}{2k}.$$

Now, by uniqueness of the zero solution, it follows that for any x_0 such that $|x_0| < \frac{1}{2k-1}$ the solution $x(t)$ satisfies

$$|x(t, t_0, x_0)| < \varepsilon.$$

And so

$$r(t_0, \varepsilon) = \varepsilon = \frac{1}{2k-1}.$$

Hence,

$$\lim_{\lambda \to 0^+} r(t_0 \varepsilon, \lambda) = \frac{1}{2k} \neq \frac{1}{2k-1} = r(t_0, \varepsilon, 0) = r(t_0, \varepsilon).$$

Now this holds for each $k \in \mathbb{N}$. Thus $r(t_0, \varepsilon, \lambda)$ is not continuous in λ.

Proposition 2. Suppose that the zero solution $x \equiv 0$ of (2.3) is conditionally totally stable. Suppose also that the origin is asymptotically stable for $\lambda = 0$ and completely unstable for $\lambda > 0$. Then $r(t_0, \varepsilon, \lambda)$ is not continuous in ε.

Proof. By Theorem 2 there exists $\lambda^* > 0$ and a neighborhood ϑ of the origin such that if $\lambda \in (0, \lambda^*)$ and M_λ is the largest invariant compact subset of $\vartheta \setminus \{0\}$ then (M_λ) is a family of asymptotically stable compact sets bifurcating from $\{0\}$.

Now let $\lambda > 0$ and $t_0 \in I$ be fixed.

Then, by hypothesis, the origin is completely unstable. Hence, we have for some $\eta > 0$ that

$$R(t_0, \lambda) = \eta.$$

Note, by definition of complete instability, one has for every $\varepsilon < \eta$

$$r(t_0, \varepsilon, \lambda) = 0.$$

Now since M_λ is an invariant set, we see by the fact that $R(t_0, \lambda) = \eta$ that for some $\alpha > 0$ and for all $\varepsilon > \eta$ $r(t_0, \varepsilon, \lambda) \geq \alpha > 0$.

Hence,

$$\lim_{\varepsilon \to R(t_0, \lambda)^+} r(t_0, \varepsilon, \lambda) \geq \alpha \neq 0 = r(t_0, R(t_0, \lambda), \lambda).$$

And so $r(t_0, \varepsilon, \lambda)$ is not continuous in ε.

Proposition 3. Assume that the zero solution of (2.3) is conditionally totally stable. Suppose also that the zero solution of (2.3) is not a uniform attractor. Then $r(t_0, \varepsilon, \lambda)$ need not be continuous in ε.

The following example is used to depict Proposition 3.

Example 2. Let $x' = f(x)$ where

$$f(x)=\begin{cases}(-1)^n\left(-x+\dfrac{1}{n+1}\right) & x\in\left[\dfrac{1}{n+1},\dfrac{2n+1}{2n(n+1)}\right)\\ (-1)^n\left(x-\dfrac{1}{n}\right) & x\in\left[\dfrac{2n+1}{2n(n+1)},\dfrac{1}{n}\right)\\ 0 & x=\dfrac{1}{n}\\ -x & x\le 0\end{cases}. \tag{3.3}$$

Note that for each $x_0=\frac{1}{n}$ the solution $x(t)$ satisfies

$$x(t,t_0,x_0)\equiv\frac{1}{n}$$

and so $x\equiv 0$ is not asymptotically stable. Let the perturbation (2.2) of (2.1) be given by

$$x_p'=f(x_p)+\lambda \tag{3.4}$$

where $\lambda>0$ is chosen such that for all x_p *with* $x_p\in\left[-\frac{1}{8},\frac{1}{8}\right]$ one has

$$x_p'\ge 0$$

and for $|x_p|=\frac{1}{8}$

$$x_p'=0.$$

Moreover, there exists a neighborhood $\left(\frac{1}{8},\bar{x}\right)$ where one has for each $x_p\in\left(\frac{1}{8},\bar{x}\right)$

$$x_p'<0.$$

Note that $g(0,\lambda)=\lambda\ne 0$. Now, choose $\varepsilon_1=\frac{1}{8}$ and *let* $\varepsilon_2<\frac{1}{8}$. Then by uniqueness of solutions and since the solution through $x_{p,0}=\frac{1}{8}$ satisfies

$$x_p\left(t,t_0,\frac{1}{8}\right)\equiv\frac{1}{8}$$

it follows that

$$r(t_0,\varepsilon_1,\lambda)=\frac{1}{8}.$$

Also,

$$r(t_0,\varepsilon_2,\lambda)=0.$$

This is due to the fact that the zero solution of the perturbed differential equation (3.4) is completely unstable and the ball of radius ε_2 intersects this neighborhood of complete instability. Hence,

$$\lim_{\varepsilon_2\to\varepsilon_1} r(t_0,\varepsilon_2,\lambda)=0\ne r(t_0,\varepsilon_1,\lambda).$$

Now this holds for each $t_0 > 0$ and $\lambda > 0$. Hence, $r(t_0, \varepsilon, \lambda)$ is not continuous in ε.

The next property which is presented again shows that the radius of stability need not be continuous in ε.

Proposition 4. Assume that the zero solution of (2.3) is conditionally totally stable. Suppose also that the zero solution of (2.4) is stable. Then $r(t_0, \varepsilon, \lambda)$ need not be continuous in ε.

An example is given to depict Proposition 4.

Example 3. Let

$$x' = f(x) = -x. \tag{3.5}$$

Construct for each $n \in \mathbb{N}$ a function $g_n(x)$ such that

$$g_n(x) = \begin{cases} 0 & x \le \frac{1}{n} \\ 4\left(x - \frac{1}{n}\right) & x \in \left(\frac{1}{n}, \frac{2}{n}\right] \\ 2x & x \in \left(\frac{2}{n}, \frac{3}{n}\right]. \\ -6\left(x - \frac{4}{n}\right) & x \in \left(\frac{3}{n}, \frac{4}{n}\right] \\ 0 & x > \frac{4}{n} \end{cases}$$

Then it follows that the zero solution $x_p \equiv 0$ of the perturbed differential equation given by

$$x_p' = f(x_p) + g_n(x_p)$$

is stable. Also, the zero solution $x \equiv 0$ of (3.5) is uniformly asymptotically stable. Now, let $\varepsilon_1 = \frac{24}{7n}$ and let $\varepsilon_2 < \frac{24}{7n}$. Then

$$r(t_0, \varepsilon_2, \lambda_n) \le \frac{1}{n}.$$

This is due to the fact that for some $x_p \in \left(\frac{1}{n}, \frac{4}{n}\right)$ one has

$$x_p' > 0.$$

Note that for $x = \frac{24}{7n}$

$$f(x_p) + g_n(x_p) \equiv 0.$$