Ronald N. Bracewell

The Fourier Transform and its Applications

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Second Edition

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McGraw-Hill Book Company
New York St. Louis San Francisco Auckland Bogotá Düsseldorf
Johannesburg London Madrid Mexico Montreal New Delhi
Panama Paris São Paulo Singapore Sydney Tokyo Toronto

The Fourier Transform and Its Applications

1234567890 DODO 78321098

This book was set in Scotch Roman by Bi-Comp, Incorporated.
The editors were Julienne V. Brown and Michael Gardner;
the production supervisor was Dennis J. Conroy.
The drawings were done by J & R Services, Inc.
R. R. Donnelley & Sons Company was printer and binder.

Library of Congress Cataloging in Publication Data

Bracewell, Ronald Newbold, date
The Fourier transform and its applications.

(McGraw-Hill electrical and electronic engineering series)
Includes index.
1. Fourier transformations. 2. Transformations (Mathematics) 3. Harmonic analysis. I. Title.
QA403.5.B7 1978 515'.723 77-13376
ISBN 0-07-007013-X

Preface to the Second Edition

The unifying role played by the Fourier transform in linking together the diverse fields mentioned in the original preface has now firmly established transform methods at the heart of the electrical engineering curriculum. Computing and data processing, which have emerged as large curricular segments, though rather different in character from circuits, electronics, and waves, nevertheless do share a common bond through the Fourier transform. Consequently, the subject matter has easily moved into the pivotal role foreseen for it, and faculty members from various specialties have found themselves comfortable teaching it. The course is taken by first-year graduate students, especially students arriving from other universities, and increasingly by students in the last year of their bachelor's degree.

Introduction of the fast Fourier transform (FFT) algorithm has greatly broadened the scope of application of the Fourier transform to data handling and has brought prominence to the discrete Fourier transform (DFT). The technological revolution brought about by these topics, now treated in a new Chapter 18, is only beginning to be felt, but will make an understanding of Fourier notions (such as aliasing, which only aficionados knew about) indispensable to any engineer who handles masses of data. In the future this will mean nearly everyone.

Transforms presented graphically in the Pictorial Dictionary proved to be a useful reference feature and have been added to. Graphical presentation adds a new dimension to the published compilations of integral transforms where it is sometimes frustrating to seek commonly needed entires among the profusion of rare cases. In addition, simple functions that are impulsive, discontinuous, or defined piecewise may not be included or may be hard to recognize.

A good problem assigned at the right stage can be extremely valuable for the student, but a good problem is hard to compose. Among the collection of supplementary problems now included at the end of the book are several that go beyond being mathematical exercises by inclusion of technical background or by asking for opinions.

Notation is a vital adjunct to thinking and I am happy to report that the *sinc function*, which we learned from P. M. Woodward's book, is alive and well and surviving erosion by occasional authors who do not know that "sine x over x" is not the sinc function. The unit rectangle function (unit height and width) $\Pi(x)$, the transform of the sinc function, has also proved extremely useful, especially for blackboard work. In typescript or other circumstances where the Greek letter is less desirable, $\Pi(x)$ may be written "rect x," and it is convenient in any case to pronounce it *rect*. The shah function $\text{III}(x)$ has caught on. It is easy to type and is twice as useful as you might think because it is its own transform. The asterisk for convolution, which was in use a long time ago by Volterra and perhaps earlier, is now in wide use and I recommend ** to denote two-dimensional convolution, which has come into extensive use as a result of the explosive growth of image processing.

Early emphasis on convolution in a text on Fourier transforms turned out to be exactly the right way to go. Convolution has changed in a few years from being presented as a rather advanced concept to one that can easily be explained at an early stage as is fitting for an operation that describes all those systems that respond sinusoidally when you shake them sinusoidally.

Ronald N. Bracewell

Preface to the First Edition

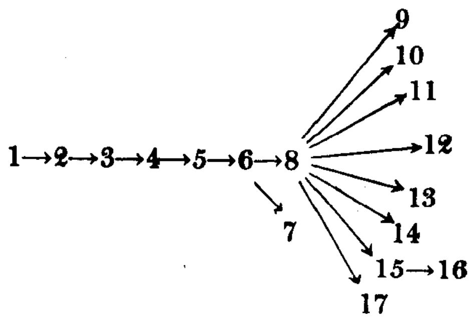
Transform methods provide a unifying mathematical approach to the study of electrical networks, devices for energy conversion and control, antennas, and other components of electrical systems, as well as to complete linear systems whether electrical or not. These same methods apply equally to the subjects of electrical communication, radio propagation, and ionized media—which are concerned in the interconnection of electrical systems—and to information theory which, among other things, relates to the acquisition, processing, and presentation of data. Other theoretical techniques are used in handling these basic fields of electrical engineering, but transform methods are virtually indispensable in all of them.

A course on transforms and their applications has formed part of the electrical engineering curriculum at Stanford University for some years, and has been given with no prerequisites which the holder of a bachelor's degree does not normally possess. One objective has been to develop a pivotal course to be taken at an early stage by all graduates, so that in later, more specialized courses, the student would be spared encountering the same basic material over and over again, and the instructor would be able to proceed more directly to his special subject matter.

It is clearly not feasible to give the whole of linear mathematics in a single course, and the choice of material must necessarily remain a matter of judgment. The choice must, however, be sharply defined if later instructors are to rely on it.

An early-level course should be simple, but not trivial; the objective of this book is to simplify the presentation of many key topics that are ordinarily dealt with in advanced contexts by making use of suitable notation and an approach through convolution.

The organization of chapters is as follows:



One way of working from the book is to take the chapters in numerical order. This sequence is feasible for students who can read the first half unassisted, or who can be taken through it rapidly in a few lectures, but if the material is approached at a more normal pace, then as a practical teaching matter, it is desirable to illustrate the theorems and concepts by dealing simultaneously with a physical topic, such as waveforms and their spectra (Chapter 9), for which the student already has some feeling.

The amount of material is suitable for one semester, or for one quarter, according to how many of the later chapters on applications are included.

Many fine mathematical texts on the Fourier transform have been published. This book differs in that it is intended for those who are concerned with applying Fourier transforms to physical situations rather than with furthering the mathematical subject as such. The connections of the Fourier transform with other transforms are also explored, and the text has been purposely enriched with condensed information that will suit it for use as a reference source for transform pairs and theorems pertaining to transforms.

My interest in the subject was fired when I was studying analysis from Carslaw's "Fourier Series and Integrals" at the University of Sydney in 1939; more recently I have applied transform methods to various problems arising in connection with directive antennas, a subject that is touched on only briefly in this book, but which may be followed up by reference to the "Encyclopedia of Physics" (vol. 54, S. Flügge, ed., Springer-Verlag, Berlin, 1962). In solving these problems I benefited from the physical approach to the Fourier transformation that I learned from J. A. Ratcliffe at the Cavendish Laboratory, Cambridge.

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Chapter 1 Introduction

Linear transforms, especially those named for Fourier and Laplace, are well known as providing techniques for solving problems in linear systems. Characteristically one uses the transformation as a mathematical or physical tool to alter the problem into one that can be solved. This book is intended as a guide to the understanding and use of transform methods in dealing with linear systems.

The subject is approached through the Fourier transform. Hence, when the more general Laplace transform is discussed later, many of its properties will already be familiar and will not distract from the new and essential question of the strip of convergence on the complex plane. In fact, all the other transforms discussed here are greatly illuminated by an approach through the Fourier transform.

Fourier transforms play an important part in the theory of many branches of science. While they may be regarded as purely mathematical functionals, as is customary in the treatment of other transforms, they also assume in many fields just as definite a physical meaning as the functions from which they stem. A waveform—optical, electrical, or acoustical—and its spectrum are appreciated equally as physically picturable and measurable entities: an oscilloscope enables us to see an electrical waveform, and a spectroscope or spectrum analyzer enables us to see optical or electrical spectra. Our acoustical appreciation is even more direct, since the ear hears spectra. Waveforms and spectra are Fourier transforms of each other; the Fourier transformation is thus an eminently physical relationship.

The number of fields in which Fourier transforms appear is surprising. It is a common experience to encounter a concept familiar from one branch of study in a slightly different guise in another. For example, the principle of the phase-contrast microscope is reminiscent of the circuit for detecting frequency modulation, and the explanation of both is conveniently given in terms of transforms along the same lines. Or a problem in statistics may yield to an approach which is familiar from studies of cascaded amplifiers. This is simply a case of the one underlying theorem from Fourier theory assuming different physical embodiments.

It is a great advantage to be able to move from one physical field to another and to carry over the experience already gained, but it is necessary to have the key which interprets the terminology of the new field. It will be evident, from the rich variety of topics coming within its scope, what a pervasive and versatile tool Fourier theory is.

Many scientists know Fourier theory not in terms of mathematics, but as a set of propositions about physical phenomena. Often the physical counterpart of a theorem is a physically obvious fact, and this allows the scientist to be abreast of matters which in the mathematical theory may be quite abstruse. Emphasis on the physical interpretation enables us to deal in an elementary manner with topics which in the normal course of events would be considered advanced.

Although the Fourier transform is vital in so many fields, it is often encountered in formal mathematics courses in the last lecture of a formidable course on Fourier series. As need arises it is introduced ad hoc in later graduate courses but may never develop into a usable tool. If this traditional order of presentation is reversed, the Fourier series then falls into place as an extreme case within the framework of Fourier transform theory, and the special mathematical difficulties with the series are seen to be associated with their extreme nature—which is nonphysical; the handicap imposed on the study of Fourier transforms by the customary approach is thus relieved.

The great generality of Fourier transform methods strongly qualifies the subject for introduction at an early stage, and experience shows that it is quite possible to teach, at this stage, the distilled theorems, which in their diverse applications offer such powerful tools for thinking out physical problems.

The present work began as a pictorial guide to Fourier transforms to complement the standard lists of pairs of transforms expressed mathematically. It quickly became apparent that the commentary would far outweigh the pictorial list in value, but the pictorial dictionary of transforms is nevertheless important, for a study of the entries reinforces the intuition, and many valuable and common types of function are included which, because of their awkwardness when expressed algebraically, do not occur in other lists.

A contribution has been made to the handling of simple but awkward functions by the introduction of compact notation for a few basic functions which are usually defined piecewise. For example, the rectangular pulse, which is at least as simple as a Gaussian pulse, is given the name $\Pi(x)$, which means that it can be handled as a simple function. The picturesque term "gate function," which is in use in electronics, suggests how a gating waveform $\Pi(t)$ opens a valve to let through a segment of a waveform. This is the way we think of the rectangle function mathematically when we use it as a multiplying factor.

Among the special symbols introduced or borrowed are $\text{II}(x)$, the even impulse pair, and $\text{III}(x)$ (pronounced *shah*), the infinite impulse train defined by $\text{III}(x) = \Sigma\, \delta(x - n)$. The first of these two gains importance from its status as the Fourier transform of the cosine function; the second proves indispensable in discussing both regular sampling or tabulation (operations which are equivalent to multiplication by *shah*) and periodic functions (which are expressible as convolutions with *shah*). Since *shah* proves to be its own Fourier transform, it turns out to be twice as useful an entity as might have been expected. Much freedom of expression is gained by the use of these conventions of notation, especially in conjunction with the asterisk notation for convolution. Only a small step is involved in writing $\Pi(x) * f(x)$, or simply $\Pi * f$, instead of

$$\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f(u)\, du$$

(for example, for the response to a photographic density distribution $f(x)$ on a sound track scanned with a slit). But the disappearance of the dummy variable and the integral sign with limits, and the emergence of the character of the response as a convolution between two profiles Π and f, lead to worthwhile convenience in both algebraic and mental manipulation.

Convolution is used a lot here. Experience shows that it is a fairly tricky concept when it is presented bluntly under its integral definition, but it becomes easy if the concept of a functional is first understood. Numerical practice on serial products confirms the feeling for convolution and incidentally draws attention to the practical character of numerical evaluation: for numerical purposes one normally prefers to have the answer to a problem come out as the convolution of two functions rather than as a Fourier transform.

This is a good place to mention that transform methods do not necessarily involve taking transforms numerically. On the contrary, some of the best methods for handling linear problems do not involve application of the Fourier or Laplace transform to the data at all; but the basis for such methods is often clarified by appeal to the transform domain. Thinking in terms of transforms, we may show how to avoid numerical harmonic analysis or the handling of data on the complex plane.

It is well known that the response of a system to harmonic input is itself harmonic, at the same frequency, under two conditions: linearity and time invariance of the system properties. These conditions are, of course, often met. This is why Fourier analysis is important, why one specifies an amplifier by its *frequency* response, why harmonic variation is ubiquitous. When the conditions for harmonic response to harmonic stimulus break down, as they do in a nonlinear servomechanism, analysis of a stimulus into harmonic components must be reconsidered. Time invariance can often be counted on even when linearity fails, but *space* invariance is by no means as common. Failure of this condition is the reason that bridge deflections are not studied by analyzing the load distribution into sinusoidal components (space harmonics).

The two conditions for harmonic response to harmonic stimulus can be restated as *one* condition: that the response shall be relatable to the stimulus by *convolution*. For work in Fourier analysis, convolution is consequently profoundly important, and such a pervasive phenomenon as convolution does not lack for familiar examples. Good ones are the relation between the distribution on a film sound track or recorder tape and the electrical signal read out through the scanning slit or magnetic head.

Many topics normally considered abstruse or advanced are presented here, and simplification of their presentation is accomplished by minor conveniences of notation and by the use of graphs.

Special care has been given to the presentation and use of the impulse symbol $\delta(x)$, on which, for example, both $\Pi(x)$ and $\text{III}(x)$ depend. The term "impulse symbol" focuses attention on the status of $\delta(x)$ as something which is not a function; equations or expressions containing it then have to have an interpretation, and this is given in an elementary fashion by recourse to a sequence of pulses (not impulses). The expression containing the impulse symbol thus acquires meaning as a limit which, in many instances, "exists." This commonplace mathematical status of the *complete expression*, in contrast with that of $\delta(x)$ itself, directly reflects the physical situation, where Green's functions, impulse responses, and the like are often accurately producible or observable, within the limits permitted by the resolving power of the measuring equipment, while impulses themselves are fictitious. By deeming all expressions containing $\delta(x)$ to be subject to special rules, we can retain both rigor and the direct procedures of manipulating $\delta(x)$ which have been so successful.

Familiar examples of this physical situation are the moment produced by a point mass resting on a beam and the electric field of a point charge. In the physical approach to such matters, which have long been imbedded in physics, one thinks about the effect produced by smaller and smaller but denser and denser massive objects or charged volumes, and notes whether the effect produced approaches a definite limit. Ways of representing this mathematically have been tidied up to the satisfaction of mathematicians only in recent years, and use of $\delta(x)$ is now licensed if accompanied by an appropriate footnote reference to this recent literature, although in some conservative fields such as statistics $\delta(x)$ is still occasionally avoided in favor of distinctly more awkward Stieltjes integral notation. These developments in mathematical ideas are mentioned in Chapters 5 and 6, the presentation in terms of "generalized functions" following Temple and Lighthill being preferred. The validity of the original physical ideas remains unaffected, and one ought to be able, for example, to discuss the moment of a point mass on a beam by considering those rectangular distributions of pressure that result from restacking the load uniformly on an ever-narrower base to an ever-increasing height; it should not be necessary to limit attention to pressure distributions possessing an infinite number of continuous derivatives merely because in some other problem a derivative of high order is involved. Therefore the subject of impulses is introduced with the aid of the rather simple mathematics of rectangle functions, and in the relatively few cases where the first derivative is wanted, triangle functions are used instead.