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Infinite Dimensional Linear Control Systems

The Time Optimal and Norm Optimal Problems

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THE TIME OPTIMAL AND NORM OPTIMAL PROBLEMS

NORTH-HOLLAND MATHEMATICS STUDIES 201

(Continuation of the Notas de Matemática)

Editor: Jan van Mill
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To
Natalia
María Elena
Sonia, José María, Cristina and Carlos
Susana and David

PREFACE

One of the first infinite dimensional control systems to come under scrutiny was

$$y'(t) = Ay(t) + u(t)\,, \quad y(0) = \zeta \tag{1}$$

where $y(t)$ takes values in a Banach space E and A is the infinitesimal generator of a strongly continuous semigroup $S(t)$. Research on the time optimal problem started in the early sixties and branched into other optimal control problems. However, many basic questions were unsolved at the end of the last century. Since then, some new results have clarified the panorama but the subject is still in need of proofs of, or counterexamples to many natural conjectures and it remains a live research area with many actual and potential applications.

In spite of being, in a sense, the simplest infinite dimensional control system, the equation (1) models some important control processes such as those described by the parabolic equation

$$\frac{\partial y(t,x)}{\partial t} = Ay(t,x) + u(t,x) \tag{2}$$

where A is an elliptic operator in the space variables $x = (x_1, x_2, \ldots, x_m)$ in a domain Ω of m-dimensional Euclidean space $I\!R^m$; the domain of A is restricted by boundary conditions. The *control* $u(t,x)$ satisfies bounds of the type

$$\int_\Omega |u(t,x)|^p dx \le C \tag{3}$$

for some p, $1 \le p < \infty$, or

$$|u(t,x)| \le C\,. \tag{4}$$

These bounds determine the *state space* E in which (2) is modeled. For the bound (3) the space is $E = L^p(\Omega)$. For the uniform bound (4) we take $E = L^\infty(\Omega)$, (or, rather $E = C(\overline{\Omega})$). The most physically significant cases are (4) for heat processes and (3) with $p = 1$ for diffusions. On the other hand, $p = 2$ leads to the simplest mathematics since the state space $L^2(\Omega)$ is a Hilbert space.

We consider two optimal control problems for the equation (1). Both of them include a *target condition*

$$y(T) = \bar{y}\,. \tag{5}$$

In the *norm optimal problem* we minimize

$$\operatorname*{ess.\,sup}_{0\le t\le T} \|u(t)\|$$

among all solutions of (1) satisfying the initial condition $y(0) = \zeta$ and the target condition (5); the control interval $0 \le t \le T$ is fixed. In the *time optimal* problem the controls satisfy a fixed bound such as

$$\operatorname*{ess.\,sup}_{0\le t\le T} \|u(t)\| \le 1\,,$$

and we minimize T subject to the initial and target condition.

The time optimal problem received privileged attention from the very start of control theory, but this has been less the case for the norm optimal problem. It was known for a long time that time optimality implies norm optimality, but that the two problems are far from equivalent in the infinite dimensional setting seems to have been realized much more recently. However, there are many situations (determined by conditions on the semigroup $S(t)$ or on the target $\bar{y}$) where time and norm optimality are essentially equivalent.

Most of this book deals with the relation among time and norm optimality and *Pontryagin's maximum principle*

$$\langle S(T-t)^* z, \bar{u}(t)\rangle = \max_{\|u\|\le 1} \langle S(T-t)^* z, u\rangle \tag{6}$$

($\|u\| \le$ minimum norm for the norm optimal problem). The maximum principle with $z \in E^* =$ dual of E is a necessary and (almost) sufficient condition for time and norm optimality in finite dimension.[1] The finite dimensional theory extends to the equation (1) when $S(t)E = E$ for $t > 0$ (in particular when $S(t)$ is a group) but the similarities with the finite dimensional case end here. In general, special assumptions on the target $\bar{y}$ are needed to make (6) a necessary condition for optimality (with z in a space larger than E^*) and, conversely, special assumptions on z are needed to make (6) a sufficient condition for optimality of $\bar{u}(t)$. In fact, *singular* optimal controls (those that do *not* satisfy Pontryagin's maximum principle, or satisfy it only in a weak form) are the main actors in various places of this monograph.

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[1] Even in finite dimension, sufficiency of the maximum principle for time optimality requires additional conditions on the initial condition ζ and/or the target $\bar{y}$.

Much of the material is independent of the maximum principle. Under suitable conditions on z, (6) implies the *bang-bang principle*

$$\|u(t)\| = 1 \quad \text{a. e.} \tag{7}$$

but (7) can be also be proved without intercession of the maximum principle for time optimal controls. Other results (some depending on the maximum principle, some not) include various well posedness properties of control problems, that is, continuous dependence of optimal controls on parameters of the system such as the initial and target conditions.

This monograph is organized as follows. The first two sections of Chapter 1 contain a survey of some finite dimensional results with an outline of infinite dimensional systems in the third. The aim is reveal that some infinite dimensional results are descendants of finite dimensional theorems. In some cases, however the "family resemblance" is slight, and many other results have no counterpart in finite dimension. We have included some references to the early history of infinite dimensional control theory in **1.3**.

Chapter 2 and Chapter 3 deal with the system (1) in an arbitrary Banach space E with a view towards the modeling of partial differential equations such as (2) in $L^p(\Omega)$ for $1 < p < \infty$. However, we do also other equations; for instance, some of the most interesting examples in **2.6** and **2.7** use the "proto-hyperbolic" equation $y_t(t,x) = -y_x(t,x)$. These results suggest that a systematic study of the maximum principle (or, rather, of its interpretation) for equations of hyperbolic type would be worth undertaking, but this is not attempted here.

Chapter 4 is on the modeling of the equation (2) in $C(\overline{\Omega})$. Due to existence requirements, the control space must be expanded to $L^\infty(\Omega)$ and we can request only weak measurability of the controls with respect to t; this corresponds to driving (2) with controls in $L^\infty((0,T) \times \Omega)$.

Chapter 5 is on the modeling of the equation (2) in $L^1(\Omega)$. The control space $L^1((0,T) \times \Omega)$ places us in a adverse existence situation, thus we must replace it by a space of (weakly measurable) controls taking values in the space $\Sigma(\overline{\Omega})$ of Borel measures in $\overline{\Omega}$.

There is an obvious parallelism between the two models in Chapters 4 and Chapter 5, so much so that one can translate results from one case to the other using a "replacement chart" where $C(\overline{\Omega})$ is replaced by $L^1(\Omega)$, $L^\infty(\Omega)$ is replaced by $\Sigma(\overline{\Omega})\ldots$ and so on. However, the similarity does not go all the way. Although both the geometries of $L^\infty(\Omega)$ and $\Sigma(\Omega)$ are devoid of smoothness, the control space $L^\infty(\Omega)$ allows uniqueness results for optimal controls not unlike those in smooth spaces, while in the space $\Sigma(\overline{\Omega})$ uniqueness breaks down completely both in the time optimal and the norm optimal problems. In a sort of compensation, in the $\Sigma(\overline{\Omega})$ setting the bang-bang principle (7) is, in certain situations, essentially sufficient for both time optimality and norm optimality, a result that is not known to hold in any other space.

The treatments in Chapters 4 and 5 could have been unified by means of the theory of Phillips adjoints, but the gain in brevity and conciseness would not outweigh the additional insight that each of the parallel theories afford.

The first two sections of Chapter 6 deal with some results that are known to hold for a very restricted class, self adjoint semigroups in Hilbert spaces, and examine the possibility of generalizations to other semigroups and Banach spaces. In the last section we include some recent references on the time and norm optimal problems as well as on problems not treated in this book but related to the material in one way or another, among them minimization of functionals other than time or norm (in particular, the linear-quadratic problem) and a sketch of the methods used in the control theory of semilinear equations. Some of these methods lean on the linear theory.

Infinite dimensional control theory is increasingly becoming the calculus of variations of the new century. We believe this monograph will have something to say to specialists (even if in a restricted area). On the other hand, the book is also accessible to beginners; the only advanced prerequisite is a course in basic linear functional analysis including semigroup theory.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Finite dimensional systems: the maximum principle. Half a century ago, Bellman, Glicksberg and Gross [1956] figured out the time optimal controls $\bar{u}(t)$ for the linear, time invariant ordinary differential system

$$y'(t) = Ay(t) + Bu(t)\,, \quad y(0) = \zeta \tag{1.1.1}$$

where A (resp. B) is a $n \times n$ matrix (resp. a $m \times n$ matrix). *Controls* $u(t)$ are measurable $\mathbb{R}^m$-valued vector functions $u(t) = (u_1(t), \dots, u_m(t))$ and a control is *admissible* if

$$\|u(t)\| \le 1 \quad \text{a. e. in} \quad 0 \le t \le T \tag{1.1.2}$$

where $\|\cdot\|$ is the *maximum norm*

$$\|u\| = \max(|u_1|, \dots, |u_m|)\,. \tag{1.1.3}$$

In the *time optimal problem* we drive the initial condition $\zeta \in \mathbb{R}^n$ to a target $\bar{y} \in \mathbb{R}^n$ in minimum time T by means of an admissible control $u(t)$; "driving to a target $\bar{y}$ in time T" means the *target condition*

$$y(T, \zeta, u) = \bar{y} \tag{1.1.4}$$

must be satisfied, where $y(t, \zeta, u)$ is the solution $y(t)$ of (1.1.1). In the *norm optimal* problem, the controls are m-vector functions with measurable, essentially bounded components, and one drives ζ to the target $\bar{y}$ in a fixed time interval $0 \le t \le T$, so that (1.1.4) is satisfied; the objective is to minimize the norm

$$\|u(\cdot)\|_{L^\infty(0,T;\mathbb{R}^m)} = \operatorname*{ess.\,sup}_{0 \le t \le T} \|u(t)\|\,. \tag{1.1.5}$$

It was proved in Bellman *et al.* [1956] that every time optimal control $\bar{u}(t)$ satisfies

$$\langle B^*S(T-t)^*z, \bar{u}(t)\rangle = \max_{\|u\| \le 1} \langle B^*S(T-t)^*z, u\rangle \tag{1.1.6}$$

in the control interval $0 \le t \le T$ = optimal time, where $S(t) = e^{tA}$ and the *multiplier* $z \in \mathbb{R}^n$ is not zero. This result was immediately submerged by *Pontryagin's* *maximum principle*, announced by Boltyanski, Gamkrelidze and Pontryagin in [1956]. The maximum principle could deal with nonlinear equations and general cost functionals and included the linear time-invariant result as a very particular case. However, the proof in Bellman *et al.* [1956] was so elementary that invited generalizations to infinite dimensional spaces, these generalizations motivated by the modeling of optimal problems for partial differential equations (for more on this see **1.3**). Here is the proof of (1.1.6), which holds as well for the norm optimal problem.

Theorem 1.1.1. *(a) Let $\bar{u}(t)$ be time optimal in the interval $0 \le t \le T$. Then the maximum principle* (1.1.6) *holds for a multiplier $z \neq 0$. (b) Let $\bar{u}(t)$ be norm optimal in the interval $0 \le t \le T$. Then $\bar{u}(t)$ satisfies the maximum principle*

$$\langle B^* S(T-t)^* z, \bar{u}(t) \rangle = \max_{\|u\| \le \rho} \langle B^* S(T-t)^* z, u \rangle \tag{1.1.7}$$

for a multiplier $z \neq 0$, where

$$\rho = \|\bar{u}(\cdot)\|_{L^\infty(0,T;\mathbb{R}^m)} \,. \tag{1.1.8}$$

Proof. Solutions of (1.1.1) are given by the variation-of-constants formula[1]

$$y(t) = y(t, \zeta, u) = S(t)\zeta + \int_0^t S(t-\sigma)Bu(\sigma)d\sigma \,. \tag{1.1.9}$$

The *reachable space* $R^\infty(T)$ of (1.1.1) is the subspace of $\mathbb{R}^n$ of all elements of the form

$$y = \int_0^T S(T-\sigma)Bu(\sigma)d\sigma \tag{1.1.10}$$

with $u(\cdot) \in L^\infty(0,T;\mathbb{R}^m)$ (that is, the space of all $y \in \mathbb{R}^n$ to which we can drive from $\zeta = 0$ with controls in $L^\infty(0,T;\mathbb{R}^m)$) and $B_\rho^\infty(T)$ is the subset of $R^\infty(T)$ of all elements of the form (1.1.10) with[2]

$$\|u(t)\| \le \rho \quad \text{a. e. in } 0 \le t \le T \,. \tag{1.1.11}$$

Clearly, $B_\rho^\infty(T)$ is a convex subset of $\mathbb{R}^n$ containing the origin. To show *(b)*, assume that $\bar{u}(t)$ is norm optimal, and let ρ be given by (1.1.8) (that is, let ρ be the optimal norm, which means the norm of the optimal control). The target condition is

$$S(T)\zeta + \int_0^T S(T-\sigma)B\bar{u}(\sigma)d\sigma = \bar{y} \,,$$

[1] Due to the controls being merely measurable, solutions are no more than absolutely continuous; the derivative $y'(t)$ exists a. e. and satisfies (1.1.1) a. e.

[2] Note that (1.1.10) will hold for many $u(\cdot)$; only one has to satisfy (1.1.11).

so that

$$\bar{y} - S(T)\zeta = \int_0^T S(T-\sigma)B\bar{u}(\sigma)d\sigma \in B_\rho^\infty(T)\,.$$

Moreover, $\bar{y} - S(T)\zeta$ is a *boundary point* of $B_\rho^\infty(T)$. In fact, if it were an interior point, since $0 \in B_\rho^\infty(T)$ we would have

$$r(\bar{y} - S(T)\zeta) \in B_\rho^\infty(T)$$

for some $r > 1$. This means

$$r(\bar{y} - S(T)\zeta) = \int_0^T S(T-\sigma)Bu(\sigma)d\sigma$$

for some control $u(t)$ satisfying the constraint (1.1.11) or, equivalently,

$$\bar{y} = S(T)\zeta + \int_0^T S(T-\sigma)B\frac{u(\sigma)}{r}d\sigma\,,$$

hence we can drive from ζ to $\bar{y}$ in time T with a control $u(\sigma)/r$ whose norm is

$$\left\|\frac{u(\cdot)}{r}\right\|_{L^\infty(0,T;\mathbb{R}^m)} \le \frac{\rho}{r} < \rho\,.$$

This contradicts the norm optimality of $\bar{u}(t)$. Having proved that $\bar{y} - S(T)\zeta$ is a boundary point of $B_\rho^\infty(T)$, we can separate it from $B_\rho^\infty(T)$ by means of a nonzero $z \in \mathbb{R}^n$; this means

$$\langle z, y\rangle \le \langle z, \bar{y} - S(T)\zeta\rangle \tag{1.1.12}$$

for every $y \in B_\rho^\infty(T)$, or

$$\left\langle z, \int_0^T S(T-\sigma)Bu(\sigma)d\sigma\right\rangle \le \left\langle z, \int_0^T S(T-\sigma)B\bar{u}(\sigma)d\sigma\right\rangle \tag{1.1.13}$$

for every $u(\cdot)$ satisfying (1.1.11). This inequality can be written

$$\int_0^T \langle z, S(T-\sigma)Bu(\sigma)\rangle d\sigma \le \int_0^T \langle z, S(T-\sigma)B\bar{u}(\sigma)\rangle d\sigma$$

or

$$\int_0^T \langle B^*S(T-\sigma)^*z, u(\sigma)\rangle d\sigma \le \int_0^T \langle B^*S(T-\sigma)^*z, \bar{u}(\sigma)\rangle d\sigma \tag{1.1.14}$$

which is equivalent to (1.1.7). This ends the proof of (b).

The proof of (a) is similar. Again, $\bar{y} - S(T)\zeta \in B_1^\infty(T)$ and to do the separation argument we need to show that $\bar{y} - S(T)\zeta$ is a boundary point of

$B_1^\infty(T)$. If $B_1^\infty(T)$ does not contain interior points (equivalently, if the inclusion $R^\infty(T) \subseteq I\!R^n$ is strict) there is nothing to prove, so we may assume that

$$R^\infty(T) = I\!R^n\,. \tag{1.1.15}$$

Property (1.1.15) is independent of $T > 0$; in fact $R^\infty(t)$ is independent of t. To see why, let $t > 0$ and assume there exists $z \in I\!R^n$ such that $\langle z, y\rangle = 0$ for all $y \in R^\infty(t)$. This is equivalent to

$$\begin{aligned}\int_0^t \langle B^*S(t-\sigma)^*z, u(\sigma)\rangle d\sigma &= \int_0^t \langle z, S(t-\sigma)Bu(\sigma)\rangle d\sigma \\ &= \left\langle z, \int_0^t S(t-\sigma)Bu(\sigma)d\sigma\right\rangle = 0\end{aligned}$$

for all $u(\cdot) \in L^\infty(0,t;I\!R^m)$, which is in turn equivalent to $B^*S(t-\sigma)^*z = 0$ or $B^*S(\sigma)^*z = 0$ for $0 \le \sigma \le t$. Now, $B^*S(\sigma)^*z$ is analytic, hence this statement is independent of t. In particular, (1.1.15) is equivalent to

$$B^*S(\sigma)^*z = 0 \quad (\sigma \ge 0) \quad \Longrightarrow \quad z = 0\,. \tag{1.1.16}$$

Assuming that (1.1.15) (or, equivalently, (1.1.16)) holds, we check that, for any $t > 0$ the expression

$$\|y\|_{R^\infty(t)} = \inf\left\{\|u(\cdot)\|_{L^\infty(0,t;I\!R^m)}\,;\ \int_0^t S(t-\sigma)Bu(\sigma)d\sigma = y\right\} \tag{1.1.17}$$

defines a norm in $R^\infty(t) = I\!R^n$, which (as any other norm) must be equivalent to the norm (1.1.3) of $I\!R^n$. In particular, this means there exists a constant $C(t)$ such that, for every $y \in I\!R^n$ we can find $u(\cdot) \in L^\infty(0,t;I\!R^m)$ with

$$y = \int_0^t S(t-\sigma)Bu(\sigma)d\sigma, \quad \|u(\cdot)\|_{L^\infty(0,t;I\!R^m)} \le C(t)\|y\|\,. \tag{1.1.18}$$

Let $t' > t$. Then we have

$$\int_0^t S(t-\sigma)Bu(\sigma)d\sigma = \int_{t'-t}^{t'} S(t'-\sigma)Bu(\sigma-(t'-t))d\sigma\,,$$

thus, if

$$y = \int_0^{t'} S(t'-\sigma)Bv(\sigma)d\sigma, \quad \|v(\cdot)\|_{L^\infty(0,t';I\!R^m)} \le C(t')\|y\|$$

is the version of (1.1.18) for t', we have

$$C(t') \le C(t) \quad (t' \ge t)\,. \tag{1.1.19}$$

Assume $\bar{y}-S(T)\zeta$ is not a boundary point of $B_1^\infty(T)$. Then, arguing as in the proof of (b) we deduce that there exists $u(\cdot) \in L^\infty(0,T;\mathbb{R}^m)$, $\|u(\cdot)\|_{L^\infty(0,T;\mathbb{R}^m)} \le r < 1$ such that

$$\bar{y} - S(T)\zeta = \int_0^T S(T-\sigma)Bu(\sigma)d\sigma \,. \tag{1.1.20}$$

Take $t < T$ and rewrite (1.1.20) as

$$\begin{aligned}\bar{y} - S(t)\zeta &= \int_0^t S(t-\sigma)Bu(\sigma)d\sigma - S(t)\zeta + S(T)\zeta \\ &\quad + \int_0^t (S(T-\sigma) - S(t-\sigma))Bu(\sigma)d\sigma + \int_t^T S(T-\sigma)Bu(\sigma)d\sigma \\ &= \int_0^t S(t-\sigma)Bu(\sigma)d\sigma + y(t,T)\,. \end{aligned} \tag{1.1.21}$$

We use the fact that $y(t,T) \to 0$ as $t \to T$, take t sufficiently near T and use (1.1.18) to construct a control $v(\cdot)$ such that $\|v(\cdot)\|_{L^\infty(0,t;\mathbb{R}^m)} \le 1 - r$ and[3]

$$\int_0^t S(t-\sigma)Bv(\sigma)d\sigma = y(t,T)\,. \tag{1.1.22}$$

Putting together (1.1.21) and (1.1.22) we obtain

$$\bar{y} = S(t)\zeta + \int_0^t S(T-\sigma)B(u(\sigma)+v(\sigma))d\sigma\,,$$

so we can drive from ζ to $\bar{y}$ by means of the admissible control $u(\cdot)+v(\cdot)$ in time $t < T$. This contradicts the fact that T is the optimal time, and ends the proof of (a), always assuming that (1.1.15) holds.

The case where $R^\infty(T) \ne \mathbb{R}^n$ is trivial; here (1.1.16) must fail, thus there exists $z \ne 0$ with $B^*S(\sigma)^*z$ identically zero. With this z every control, optimal or not, satisfies the necessary conditions (1.1.6) or (1.1.7). This completes the proof of Theorem 1.1.1.

Condition (1.1.16) and the fact that $z \ne 0$ implies that (1.1.6) is nontrivial, that is, it gives information on the optimal control $\bar{u}(t)$ for almost all t (although this information may not determine $\bar{u}(t)$ uniquely; see **1.2**).

Conditions (1.1.15) $\Leftrightarrow$ (1.1.16) also imply that the assumptions in Theorem 1.1.1 are *sufficient* (with additional conditions in the time optimal case). See Theorem 1.1.3 below.

It follows from (1.1.6) that if condition (1.1.16) is satisfied then time optimal controls satisfy not only the bound (1.1.2) but

$$\|u(t)\| = 1 \quad \text{a. e. in } 0 \le t \le T\,. \tag{1.1.23}$$

[3] Here it is essential that the constant $C(t)$ in (1.1.18) does not increase as we move t to the right; this is (1.1.19).