CONTEMPORARY MATHEMATICS

348

Inverse Problems and Spectral Theory

Proceedings of the Workshop on Spectral Theory of Differential Operators and Inverse Problems
October 28–November 1, 2002
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Kyoto University, Kyoto, Japan

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Inverse Problems and Spectral Theory

Preface

This is the proceedings of the workshop *Spectral theory of differential operators and the inverse problems* held in the Research Institute for Mathematical Science in Kyoto University from October 28 to November 1 in 2002. In all, almost 100 participants were present at RIMS during this conference, which shows the increasing interest in this field at the intersection of pure and applied mathematics. In the wide range of research on inverse problems, we tried to focus on spectral theory for differential operators and related inverse problems. The titles of the talks were as follows.

V. G. Romanov (Sobolev Inst.) : An inverse problem of electrodynamics
Y. Kurylev (Loughborough Univ.) : Inverse boundary spectral problem for the system of electromagnetism
H. Kang (Seoul National Univ.) : Asymptotic expansion for the Helmholtz equation and applications
S. Kim (Univ. of Tokyo) : Uniqueness in determining inhomogeneities of conductivity in Maxwell's equation with a single measurement
M. Yamamoto (Univ. of Tokyo) : On some uniqueness in inverse scattering problems
M. Ikehata (Gunma Univ.) : Extracting from finitely many noisy Cauchy data
J. Ralston (UCLA) : On the inverse boundary value problem of linear, isotropic elasticity
G. Nakamura (Hokkaido Univ.) : Identification of cavity in inhomogeneous media
K. Tanuma (Gunma Univ.) : Reconstruction of anisotropic elastic tensor at the boundary from the localized Dirichlet to Neumann map
S. Nakagiri (Kobe Univ.) : Constant parameters identification problems for sine-Gordon equation
M. Watanabe (Tokyo Metropolitan Univ.) : Inverse scattering problem for the nonlinear Schrödinger equations with cubic convolution nonlinearity
G. Eskin (UCLA) : Inverse boundary value problems and Aharonov-Bohm effect
E. Korotyaev (Humboldt Univ.) : Inverse problem for the harmonic oscillators
A. Melin (Lund Univ.) : Hyperbolicity and intertwining techniques in the backscattering problem
A. Vasy (M.I.T.) : Inverse problems in many-body scattering
J. Cheng (Fudan Univ.) : The numerical method for finding the discontinuous solutions of some ill-posed problems
A. Katsuda (Okayama Univ.) : Asymptotics of heat kernels for nilpotent coverings
R. Kuwabara (Tokushima Univ.) : Quantum energies and classical orbits in a

magnetic field
G. Vodev (Nantes Univ.) : High frequency estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds
G. Uhlmann (Washington Univ.) : On determining a Riemannian manifold from the length of geodesics
H. Isozaki (Tokyo Metropolitan Univ.) : Inverse spectral problems and hyperbolic manifolds
K. Onishi (Ibaraki Univ.) : Numerical method for inverse boundary value problems by the adjoint method
T. Takiguchi (National Defense Acad.) : Reconstruction of measurable plane sets from their orthogonal projections
T. Kako (Univ. of Electro Communication) : Discrete approximation of the Dirichlet-Neumann map and its applications
H. Urakawa (Tohoku Univ.) : Dirichlet eigenvalue problem, finite element method and graph theory
R. Geller (Univ. of Tokyo) : Numerical methods for efficient and accurate calculation of seismic wave propagation and applications to inverse problems
K. M. Schmidt (Cardiff Univ.) : Critical values and spectral asymptotics of singularly perturbed periodic differential operators

As one sees from the titles above, we selected topics from the following subjects which have seen intense activity recently

Electromagnetism, Elasticity, Schrödinger equation, Differential geometry, Numerical analysis.

Electromagnetism: The conference began with Romanov's talk on the inverse problem for Maxwell's equations of the uniqueness of coefficients with the given data observed on a sphere in a finite time interval. Kurylev applied the boundary control method for the reconstruction of coefficients of Maxwell's equations on a 3-manifold from given boundary spectral data. Kang stated results on the reconstruction of small diameter inclusions via boundary measurements. Kim talked about the determination of inhomogeneities in conductivity for the boundary value problem for Maxwell's equations. The inverse scattering by obstacles in 2-dimension was discussed by Yamamoto. Ikehata used Mittag-Leffer's function to extract information on inclusions (or defects) of conductivities in 2-dimensions. Nakamura talked about the reconstruction procedure for a cavity in an inhomogeneous transversally isotropic medium from the Dirichlet-to-Neumann map.

Elasticity: The inverse boundary value problem for the elastic equation is of no less importance than the Maxwell's equations. Tanuma talked about the computation of the stress tensor and its normal derivative on the boundary from the DN map. Ralston discussed the uniqueness of the Lamé parameters with the given DN map as well as the solvability of the Cauchy-Riemann systems, an essential tool in solving inverse problems for systems of PDE.

Schrödinger equation: Since this is the very source of inverse problems, both the forward and inverse problems were discussed. Nakagiri considered the estimation of coefficients in the sine-Gordon equation by cost functionals. Watanabe talked about the reconstruction of nonlinear terms from the scattering operator for the Schrödinger equation. Eskin discussed the uniqueness of the electromagnetic potentials associated with the Aharonov-Bohm effect. Korotyaev gave a characterization of the spectrum of the 1-dimensional harmonic oscillator perturbed by a potential. Melin constructed the intertwining operator for the Schrödinger operator and applied it to the back-scattering problem. Vasy talked about the inverse problem for the many-body Schrödinger equation from the various S-matrices. Schmidt gave a survey on the perturbed periodic Sturm-Liouville operator and the Dirac operator and the asymptotic distribution of its eigenvalues.

Differential geometry: Katsuda studied the decay of the heat kernels on nilpotent coverings over finite graphs or compact manifolds and asymptotic properties of closed geodesics for nilpotent extensions. Kuwabara derived a quantization condition for the magnetic Schrödinger operator on a compact Riemannian manifold. Vodev talked about high-energy estimates for the resolvent of the Laplace-Beltrami operator on non-compact Riemmanian-manifolds and its applications to the problem of resonance. Uhlmann gave a survey on the determination of Riemannian manifold from the knowledge of its geodesics. This problem has its origin in the seismology. Isozaki gave a new idea on the use of hyperbolic manifolds for solving inverse problems and its applications. Takiguchi gave a result on the reconstruction of a domain in the plane from its projections along the x and y axes.

Numerical analysis: Cheng talked about numerical differentiation by the use of Tikhonov's regularization and its application to the reconstruction of discontinuities. Onishi studied the elliptic boundary value problem with Dirichlet data on some part of the boundary and the Neumann data on another part and gave the numerical results. Kako talked about a numerical treatment of the 2-D Helmholtz equation in an unbounded domain by the domain decomposition technique and its application to voice generating phenomena. Urakawa discussed relations between the Dirichlet eigenvalue problem in a bounded planar domain, the finite element method and the graph theory. Geller's talk was on the error estimates for the numerical treatment of the elastic equation and numerical examples.

Although we could not include all of these topics, the papers in this volume present lots of new results, new ideas and comprehensive surveys of a variety of fields in inverse problems. We appreciate deeply the cooperation of Professors H. Soga, O. Yamada, M. Yamamoto, Y. Iso and M. Kawashita whose support by Grants-in-Aide made it possible to organize this conference. We also thank the referees for their careful reading and useful suggestions to the contributed papers.

Hiroshi ISOZAKI
Department of Mathematics,
Graduate School of Science,
Tokyo Metropolitan University,
Minami-Osawa, Hachioji, 192-0397, Japan

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On the determination of wave speed and potential in a hyperbolic equation by two measurements

V. G. ROMANOV† and M. YAMAMOTO‡

ABSTRACT. We discuss a problem of finding a speed of sound $c(x)$ and a potential $q(x)$ in a second-order hyperbolic equation from two boundary observations. The coefficients are assumed to be unknown inside a disc in $\mathbb{R}^2$. On a suitable bounded part of the cylindrical surface, we are given Cauchy data for solutions to a hyperbolic equation with zero initial data and sources located on the lines $\{(x,t) \in \mathbb{R}^3 \,|\, x \cdot \nu = 0, t = 0\}$ for two distinct unit vectors $\nu = \nu^{(k)}$, $k = 1, 2$. We obtain a conditional stability estimate under a priori assumptions on smallness of $c(x) - 1$ and $q(x)$.

1. Statement of the inverse problem and main results

In the papers [2], [8] - [11], a new method for obtaining conditional stability estimates for problems related to determination of coefficients for linear hyperbolic equations has been proposed. This method uses a single observation for finding one unknown coefficient.

By our method, we can prove the stability in determining coefficients by means of a finite number of measurements where initial data are zero and impulsive inputs are added. As other methodology for inverse problems with a finite number of measurements, we refer to [1], [4], [5], [7] and the references therein. In particular, the detailed proof in [1] is given for example in [5], [7]. However in those papers, we have to assume some positivity or non-degeneracy of initial values, which is not practical. For our method, we need not such restrictions on initial data, which is very practical. On the other hand, we have to assume that unknown coefficients should be close to fixed reference coefficients which are constant.

An analysis shows that the problem with several unknown coefficients under the derivatives of the first order can also be successfully studied by this method (see [9], [10]). However its application to determination of coefficients under derivatives of different orders meets some difficulties. Recently the problems of finding a damping coefficient and a potential from two measurements, and the speed of sound and damping, were considered in papers [3] and [12], respectively. In this paper, by two measurements, we consider the inverse problem where coefficients of the leading

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term and the lowest term are unknown. The technique of this paper differs from [3] and [12], but keeps some common features with them.

Let $u = u(x,t)$, $x \in \mathbb{R}^2$, satisfy the equation

$$u_{tt} - c^2(\Delta u + qu) = 2\delta(t)\,\delta(x\cdot\nu), \quad (x,t)\in\mathbb{R}^3, \tag{1.1}$$

and the zero initial condition

$$u|_{t<0} = 0. \tag{1.2}$$

Here ν is a unit vector and the symbol $x\cdot\nu$ means the scalar product of the vectors x and ν. The solution to problem (1.1) - (1.2) depends on the parameter ν, i.e., $u = u(x,t,\nu)$.

Assume that the supports of the coefficients $q(x)$ and $c(x) - 1$ are located strictly inside the disc $B := \{x\in\mathbb{R}^2|\,|x-x^0|<r\}$ and B belongs to the half-plane $x\cdot\nu > 0$. Suppose also that $q(x)$ and $c(x) > 0$ are smooth functions in $\mathbb{R}^2$ (see below).

Introduce the function $\tau(x,\nu)$ as the solution to the following problem for the eikonal equation:

$$|\nabla\tau|^2 = c^{-2}(x), \quad \tau|_{x\cdot\nu=0} = 0. \tag{1.3}$$

Let $G(\nu)$ be the cylindrical domain $G(\nu) := \{(x,t)|\,x\in B, \tau(x,\nu)<t<T+\tau(x,\nu)\}$ where T is a positive number. Denote by $S(\nu)$ the lateral boundary of this domain and by $\Sigma_0(\nu)$ and $\Sigma_T(\nu)$ the lower and upper basements, respectively. That is, $S(\nu) := \{(x,t)|\,x\in\partial B, \tau(x,\nu)\le t\le T+\tau(x,\nu)\}$, $\Sigma_0(\nu) := \{(x,t)|\,x\in B, t = \tau(x,\nu)\}$, $\Sigma_T(\nu) := \{(x,t)|\,x\in B, t = T+\tau(x,\nu)\}$, $\partial B := \{x\in\mathbb{R}^2|\,|x-x^0| = r\}$.

Consider the problem of determination of $q(x)$ and $c(x)$. Let the following information be known. We take distinct unit vectors $\nu^{(1)}$ and $\nu^{(2)}$ such that B belongs to the half plane $x\cdot\nu^{(k)} > 0$ for $k = 1,2$. Then we are given the traces of the functions $\tau(x,\nu^{(k)})$ on ∂B, and the traces on $S(\nu^{(k)}) := S_k$ of solutions and its normal derivatives to problem (1.1) - (1.2) with $\nu = \nu^{(k)}$, that is,

$$u(x,t,\nu^{(k)}) = f^{(k)}(x,t), \quad \frac{\partial}{\partial n}u(x,t,\nu^{(k)}) = g^{(k)}(x,t), \quad (x,t)\in S_k;$$
$$\tau(x,\nu^{(k)}) = \tau^{(k)}(x), \quad x\in\partial B; \quad k = 1,2. \tag{1.4}$$

The problem is: find $q(x)$ and $c(x)$ from given data, i.e., from $f^{(k)}$, $g^{(k)}$, $\tau^{(k)}$, $k = 1,2$.

For fixed constants $q_0 > 0$ and $d > 0$, let $\Lambda(q_0,d)$ be the set of functions (q,c) satisfying the following two conditions:

1) supp $q(x)$, supp $(c(x)-1)\subset\Omega\subset B$, $\mathrm{dist}(\partial B,\Omega)\ge d$,

2) $\|q\|_{\mathbf{C}^{17}(\mathbb{R}^n)}\le q_0$, $\|c-1\|_{\mathbf{C}^{19}(\mathbb{R}^n)}\le q_0$.

In particular, we note that $\nu^{(1)}$ and $\nu^{(2)}$ are linearly independent.

We prove here the following stability and uniqueness theorems.

Theorem 1.1. *Let $(q_j,c_j)\in\Lambda(q_0,d)$, and let $\{f_j^{(k)}, g_j^{(k)}, \tau_j^{(k)}\}$ be the data corresponding to the solution to (1.1) - (1.2) with $q = q_j(x)$, $c = c_j(x)$ and $\nu = \nu^{(k)}$, $k,j = 1,2$. Moreover let the condition $4r/T < 1$ be satisfied. Then there exist positive numbers q^* and C depending on T, r, d and $|\nu^{(1)} - \nu^{(2)}|$ such that for all*

$q_0 \le q^*$ the following inequality holds:

$$\|q_1 - q_2\|^2_{\mathbf{L}^2(B)} + \|c_1 - c_2\|^2_{\mathbf{H}^2(B)}$$
$$\le C \sum_{k=1}^{2} \Big(\|\widehat{f}_1^{(k)} - \widehat{f}_2^{(k)}\|^2_{\mathbf{H}^3(\partial B\times\{0\})} + \|(\widehat{f}_1^{(k)} - \widehat{f}_2^{(k)})_t\|^2_{\mathbf{H}^2(\partial B\times(0,T))} \tag{1.5}$$
$$+ \|(\widehat{g}_1^{(k)} - \widehat{g}_2^{(k)})_t\|^2_{\mathbf{H}^1(\partial B\times(0,T))} + \|\tau_1^{(k)} - \tau_2^{(k)}\|^2_{\mathbf{H}^5(\partial B)} \Big),$$

where $\widehat{f}_j^{(k)}(x,t) = f_j^{(k)}(x, t-\tau_j^{(k)}(x))$ and $\widehat{g}_j^{(k)}(x,t) = g_j^{(k)}(x, t-\tau_j^{(k)}(x))$.

Theorem 1.2. *Let the conditions the Theorem 1.1 be fulfilled. Then one can find a number $q^* > 0$ such that if $(q_j, c_j) \in \Lambda(q^*, d)$, $j = 1, 2$, and the corresponding data partly coincide, namely,*

$$f_1^{(k)}(x,t) = f_2^{(k)}(x,t),\ (x,t) \in S_k; \quad \tau_1^{(k)}(x) = \tau_2^{(k)}(x),\ x \in \partial B;\ k = 1, 2, \tag{1.6}$$

then $q_1(x) = q_2(x)$ and $c_1(x) = c_2(x)$.

Theorem 1.1 is proven in §2. To prove Theorem 1.2 we use the following assertion proven in [12] (see §4). If $u_1(x,t,\nu) = u_2(x,t,\nu)$ on $S(\nu)$ and $\tau_1(x,\nu) = \tau_2(x,\nu)$ on ∂B, then $(\nabla u_1 \cdot \mathbf{n}) = (\nabla u_2 \cdot \mathbf{n})$ on $S(\nu)$, where $\mathbf{n}$ is the outward unit normal to $S(\nu)$. Then Theorem 1.2 is a simple corollary of Theorem 1.1.

In §2 we also use the following lemma, whose proof is similar to Lemma 1.1 in [12] and we omit it here.

Lemma 1.1. *For each fixed $T_0 > 0$ there exists positive number $q^* = q^*(T_0)$ such that for $(q, c) \in \Lambda(q_0, d)$ and $q_0 \le q^*$ the solution to problem (1.1) - (1.2) in the domain $K(T_0, \nu) := \{(x,t) |\, t \le T_0 - \tau(x,\nu)\}$ can be represented in the form*

$$u(x,t,\nu) = \sum_{k=0}^{5} \alpha_k(x,\nu)\, \theta_k(t - \tau(x,\nu)) + u_5(x,t,\nu), \tag{1.7}$$

where $\theta_0(t)$ is the Heaviside function: $\theta_0(t) = 1$ for $t \ge 0$ and $\theta_0(t) = 0$ for $t < 0$, $\theta_k(t) = \frac{t^k \theta_0(t)}{k!}$, the coefficients $\alpha_k(x,\nu)$ are given in the form

$$\alpha_0(x,\nu) = \exp(\varphi(x,\nu)),\ \varphi(x,\nu) = -\frac{1}{2} \int\limits_{\Gamma(x,\nu)} c^2(\xi)\Delta\tau(\xi,\nu)\, ds,$$

$$\alpha_k(x,\nu) = \frac{\alpha_0(x,\nu)}{2} \int\limits_{\Gamma(x,\nu)} \frac{c^2(\xi)(\Delta\alpha_{k-1}(\xi,\nu) + q(\xi)\alpha_{k-1}(\xi,\nu))}{\alpha_0(\xi,\nu)}\, ds, \tag{1.8}$$
$$k = 1, \dots, m,$$

where $\Gamma(x,\nu)$ is the geodesic line joining the line $\{\xi \in \mathbb{R}^2 | \xi \cdot \nu = 0\}$ and x with respect to ds, and ds is the element of the Riemannian length: $ds = c^{-1}(x)(\sum_{k=1}^{2} dx_k^2)^{1/2}$. Then $\tau(x,\nu) \in \mathbf{C}^{19}(\Omega(T_0,\nu))$, $\alpha_k(x,\nu) \in \mathbf{C}^{17-2k}(\Omega(T_0,\nu))$ for $\Omega(T_0,\nu) := \{x \in \mathbb{R}^2 | \tau(x,\nu) \le T_0/2\}$, and the function $u_5(x,t,\nu)$ vanishes for $t \le \tau(x,\nu)$ and belongs to $\mathbf{H}^6(K(T_0,\nu))$ for fixed ν. Moreover there exists a positive number C depending on T, r and q_0 such that C does not increase as q_0 decreases and that the following inequalities hold

$$\|u - 1\|_{\mathbf{H}^6(G(\nu))} \le Cq_0, \quad \|\tau(x,\nu) - x\cdot\nu\|_{\mathbf{C}^{18}(B)} \le Cq_0. \tag{1.9}$$

Corollary. *If $(q,c) \in \Lambda(q_0,d)$ and q_0 is sufficiently small, then the function $u(x,t,\nu)$ is continuous on the closure of domain $G(\nu)$ together with all derivatives up to the fourth-order.*

2. Proof of Theorem 1.1

Introduce the function $\widehat{u}(x,t,\nu) := u(x,t+\tau(x,\nu),\nu)$. Then, by (1.1) and (1.7), the function $\widehat{u}(x,t,\nu)$ for $(x,t) \in B \times (0,T)$, satisfies

$$\begin{aligned} &2\nabla\widehat{u}_t \cdot \nabla\tau - \Delta\widehat{u} - q\widehat{u} + (\Delta\tau)\,\widehat{u}_t = 0, \quad (x,t) \in B \times (0,T); \\ &\widehat{u}(x,+0,\nu) = \alpha_0(x,\nu), \quad \widehat{u}_t(x,+0,\nu) = \alpha_1(x,\nu), \end{aligned} \tag{2.1}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$.

Substituting (1.7) into (1.1) and equating the terms of $\delta(t-\tau(x,\nu))$, $\theta_0(t-\tau(x,\nu))$, we see that the functions $\varphi(x,\nu) \equiv \ln\alpha_0(x,\nu)$ and $\alpha_1(x,\nu)$ satisfy the first-order differential equations:

$$\begin{aligned} &2\nabla\varphi \cdot \nabla\tau + \Delta\tau = 0, \\ &2\nabla\alpha_1 \cdot \nabla\tau + \alpha_1\Delta\tau - \Delta\alpha_0 - q\alpha_0 = 0. \end{aligned} \tag{2.2}$$

The latter of these equations and equation (2.1) can be rewritten respectively in the forms

$$2(\nabla\alpha_1 \cdot \nabla\tau - \alpha_1\nabla\varphi \cdot \nabla\tau) - \alpha_0(\Delta\varphi + |\nabla\varphi|^2 + q) = 0 \tag{2.3}$$

and

$$\begin{aligned} &2\nabla\widehat{u}_t \cdot \nabla\tau - \Delta\widehat{u} - 2(\nabla\varphi \cdot \nabla\tau)\widehat{u}_t - q\widehat{u} = 0, \quad (x,t) \in B \times (0,T); \\ &\widehat{u}(x,+0,\nu) = \alpha_0(x,\nu), \quad \widehat{u}_t(x,+0,\nu) = \alpha_1(x,\nu). \end{aligned} \tag{2.4}$$

Introduce $v(x,t,\nu) = \ln\widehat{u}(x,t,\nu)$ and assume that q_0 is small enough in order that the function $\widehat{u}(x,t,\nu)$ is positive in $B \times (0,T)$. The function $v(x,t,\nu)$ satisfies the relations

$$\begin{aligned} &2\nabla v_t \cdot \nabla\tau - \Delta v - |\nabla v|^2 \\ &\quad +2(\nabla v \cdot \nabla\tau - \nabla\varphi \cdot \nabla\tau)v_t - q = 0, \quad (x,t) \in B \times (0,T); \\ &\quad v(x,+0,\nu) = \varphi(x,\nu), \quad v_t(x,+0,\nu) = \beta(x,\nu), \end{aligned} \tag{2.5}$$

where $\beta(x,\nu) = \alpha_1(x,\nu)/\alpha_0(x,\nu)$ solves the equation

$$2\nabla\beta \cdot \nabla\tau - \Delta\varphi - |\nabla\varphi|^2 - q = 0. \tag{2.6}$$

Let $(q_j,c_j) \in \Lambda(q_0,d)$ for $j = 1,2$. Denote the functions u, $\widehat{u}$, v, φ, α_0, β, τ corresponding to the coefficients (q_j,c_j) by u_j, $\widehat{u}_j$, v_j, φ_j, α_{0j}, β_j, τ_j and introduce the differences

$$\begin{aligned} &\widetilde{u} = \widehat{u}_1 - \widehat{u}_2,\ \widetilde{v} = v_1 - v_2,\ \widetilde{\varphi} = \varphi_1 - \varphi_2,\ \widetilde{\alpha}_0 = \alpha_{01} - \alpha_{02}, \\ &\widetilde{\beta} = \beta_1 - \beta_2,\ \widetilde{\tau} = \tau_1 - \tau_2,\ \widetilde{c} = c_1 - c_2,\ \widetilde{q} = q_1 - q_2. \end{aligned}$$

Then we can obtain the relations

$$\begin{aligned} &2\nabla\widetilde{v}_t \cdot \nabla\tau_1 - \Delta\widetilde{v} + a_1 \cdot \nabla\widetilde{v} + a_2\,\widetilde{v}_t + a_3 \cdot \nabla\widetilde{\tau} \\ &\quad +a_4 \cdot \nabla\widetilde{\varphi} - \widetilde{q} = 0, \quad (x,t) \in B \times (0,T); \\ &\widetilde{v}(x,+0,\nu) = \widetilde{\varphi}(x,\nu), \quad \widetilde{v}_t(x,+0,\nu) = \widetilde{\beta}(x,\nu), \end{aligned} \tag{2.7}$$

where $a_1 = -\nabla(v_1+v_2)+2(v_2)_t\nabla\tau_2$, $a_2 = 2(\nabla v_1\cdot\nabla\tau_1-\nabla\varphi_1\cdot\nabla\tau_1)$, $a_3 = 2\nabla(v_2)_t+2(v_2)_t(\nabla v_1-\nabla\varphi_1)$, $a_4 = -2(v_2)_t\nabla\tau_2$. ¿From equations (2.2) and (2.6), it follows that the functions $\widetilde{\alpha}_0(x,\nu)$, $\widetilde{\beta}(x,\nu)$, $\widetilde{\varphi}(x,\nu)$ satisfy the relations

$$\begin{gathered}\widetilde{\alpha}_0 = b_1\widetilde{\varphi}, \quad \nabla\widetilde{\varphi}\cdot b_2+\nabla\widetilde{\tau}\cdot b_3+\Delta\widetilde{\tau}=0,\\ \Delta\widetilde{\varphi}+\nabla\widetilde{\varphi}\cdot h_1+\nabla\widetilde{\beta}\cdot h_2+\nabla\widetilde{\tau}\cdot h_3+\widetilde{q}=0,\end{gathered} \tag{2.8}$$

where

$$b_1 = \int_0^1 \exp[\varphi_2(1-\eta)+\varphi_1\eta]\,d\eta, \quad b_2 = \nabla(\tau_1+\tau_2), \quad b_3 = \nabla(\varphi_1+\varphi_2),$$

$$h_1 = \nabla(\varphi_1+\varphi_2), \quad h_2 = -\nabla(\tau_1+\tau_2), \quad h_3 = -\nabla(\beta_1+\beta_2).$$

Introduce the function $w(x,t,\nu) := \widetilde{v}_t(x,t,\nu)$. Then

$$\begin{gathered}2\nabla w_t\cdot\nabla\tau_1-\Delta w+a_1\cdot\nabla w+a_2\,w_t+(a_2)_t w+(a_1)_t\cdot\nabla\widetilde{v}\\ +(a_3)_t\cdot\nabla\widetilde{\tau}+(a_4)_t\cdot\nabla\widetilde{\varphi}=0, \quad (x,t)\in B\times(0,T);\\ w(x,+0,\nu)=\widetilde{\beta}(x,\nu).\end{gathered} \tag{2.9}$$

Note that the function $\widetilde{v}$ can be represented in the form

$$\widetilde{v}(x,t,\nu) \;=\; \widetilde{\varphi}(x,\nu)+\int_0^t w(x,\eta,\nu)\,d\eta, \quad (x,t)\in B\times(0,T). \tag{2.10}$$

From Lemma 1.1 and the embedding theorems, by the definition we have

$$\begin{gathered}\max_{1\le k\le 4}\|a_k\|_{\mathbf{C}^2(B\times(0,T))}\le Cq_0,\\ \max_{k=1,2}\|b_k\|_{\mathbf{C}^1(B\times(0,T))}\le C, \quad \|b_3\|_{\mathbf{C}^1(B\times(0,T))}\le Cq_0,\\ \|h_2\|_{\mathbf{C}(B\times(0,T))}\le C, \quad \max_{k=1,3}\|h_k\|_{\mathbf{C}(B\times(0,T))}\le Cq_0\end{gathered} \tag{2.11}$$

Here and henceforth $C>0$ denotes a generic constant which depends on T, r, q_0 and does not increase as q_0 decreases. Therefore relations (2.8) – (2.11) lead to the following inequalities

$$\begin{aligned}&\|\widetilde{q}\|^2_{\mathbf{L}^2(B)}\le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}+\|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)}+\|\widetilde{\beta}\|^2_{\mathbf{H}^1(B)}\right),\\ &\|\Delta\widetilde{\tau}\|^2_{\mathbf{H}^1(B)}\le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}+q_0^2\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)}\right),\\ &\|2\nabla w_t\cdot\nabla\tau_1-\Delta w\|^2_{\mathbf{H}^1(B\times(0,T))}\le Cq_0^2\left(\|w\|^2_{\mathbf{H}^2(B\times(0,T))}+\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)}+\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}\right).\end{aligned} \tag{2.12}$$

We will use the obvious inequality:

$$\|\widetilde{\tau}\|^2_{\mathbf{H}^3(B)}\le C\left(\|\Delta\widetilde{\tau}\|^2_{\mathbf{H}^1(B)}+\sum_{|\gamma|\le 3}\|D^\gamma\widetilde{\tau}\|^2_{\mathbf{L}^2(\partial B)}\right), \tag{2.13}$$

where

$$D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\partial x_2^{\gamma_n}}, \quad \gamma=(\gamma_1,\gamma_2), \quad |\gamma|=\gamma_1+\gamma_2.$$

Since $\operatorname{supp}(c(x)-1)\subset\Omega\subset B$ and $\operatorname{dist}(\partial B,\Omega)\ge d$, the function $\widetilde{\tau}(x,\nu)$ vanishes together with all its derivatives on ∂B anywhere except the set $\partial B_+(\nu) := \{x\in\partial B|\,\nu\cdot(x-x^0)>\sqrt{r^2-(r-d)^2}\}$. Moreover, since outside of B, the function $\widetilde{\tau}$

satisfies the equation $\nabla\widetilde{\tau}\cdot\nabla(\tau_1+\tau_2)=0$, all its derivatives of $\widetilde{\tau}$ on $\partial B_+(\nu)$ can be expressed via the derivatives along $\partial B_+(\nu)$. Therefore we have

$$\sum_{|\gamma|\le 3}\|D^\gamma\widetilde{\tau}\|^2_{\mathbf{L}^2(\partial B)}\le C\|\widetilde{\tau}\|^2_{\mathbf{H}^3(\partial B)}. \tag{2.14}$$

Using the second inequality in (2.12), from (2.13) and (2.14) we find

$$\|\widetilde{\tau}\|^2_{\mathbf{H}^3(B)}\le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}+q_0^2\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)}+\|\widetilde{\tau}\|^2_{\mathbf{H}^3(\partial B)}\right). \tag{2.15}$$

Consequently, for small q_0, we obtain the inequality

$$\|\widetilde{\tau}\|^2_{\mathbf{H}^3(B)}\le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}+\|\widetilde{\tau}\|^2_{\mathbf{H}^3(\partial B)}\right). \tag{2.16}$$

The following lemma is one key, which can be proved by the multiplier method similarly to Lemma 4.3.6 from [10] (see also [8]). Moreover, as a possible method for proving the lemma, we can use a Carleman estimate (e.g., [6]).

Lemma 2.1. *Let $c\in\Lambda(q_0,d)$, $4r/T<1$ and $z(x,t)\in\mathbf{H}^2(B\times(0,T))$. Then for sufficiently small q_0, there exists a positive constant C such that the following inequality holds:*

$$\begin{aligned}&\|z\|^2_{\mathbf{H}^1(B\times(0,T))}+\|z\|^2_{\mathbf{H}^1(B\times\{0\})}\\&\le C\Big(\|2\nabla z_t\cdot\nabla\tau-\Delta z\|^2_{\mathbf{L}^2(B\times(0,T))}\\&\quad+\|z\|^2_{\mathbf{H}^1(\partial B\times(0,T))}+\|\nabla z\cdot\mathbf{n}\|^2_{\mathbf{L}^2(\partial B\times(0,T))}\Big).\end{aligned} \tag{2.17}$$

Applying (2.17) with $\tau=\tau_1$ to the function $w(x,t,\nu)$ and its first derivatives and using the third inequality in (2.12), we obtain

$$\begin{aligned}&\|w\|^2_{\mathbf{H}^2(B\times(0,T))}+\|w\|^2_{\mathbf{H}^2(B\times\{0\})}\\&\le C[q_0^2(\|w\|^2_{\mathbf{H}^2(B\times(0,T))}+\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)}+\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)})+\varepsilon^2(\nu)],\end{aligned} \tag{2.18}$$

where

$$\varepsilon^2(\nu)=\|(\widehat{f}_1-\widehat{f}_2)_t\|^2_{\mathbf{H}^2(\partial B\times(0,T))}+\|(\widehat{g}_1-\widehat{g}_2)_t\|^2_{\mathbf{H}^1(\partial B\times(0,T))}. \tag{2.19}$$

and $\widehat{f}_j(x,t)=f_j(x,t-\tau_j(x,\nu))$, $\widehat{g}_j(x,t)=g_j(x,t-\tau_j(x,\nu))$, $j=1,2$.

From relation (2.18) for sufficiently small q_0, we derive the inequality

$$\|\widetilde{\beta}\|^2_{\mathbf{H}^2(B)}=\|w\|^2_{\mathbf{H}^2(B\times\{0\})}\le C[q_0^2(\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)}+\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)})+\varepsilon^2(\nu)]. \tag{2.20}$$

Then from the first inequality in (2.12), we see that

$$\|\widetilde{q}\|^2_{\mathbf{L}^2(B)}\le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}+\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)}+\varepsilon^2(\nu)\right). \tag{2.21}$$

Consider inequalities (2.16), (2.20), (2.21), the second and third relations in (2.8) for $\nu=\nu^{(k)}$, $k=1,2$. We set $\widetilde{\alpha}_0(x,\nu^{(k)})=\widetilde{\alpha}_{0k}(x)$, $\widetilde{\varphi}(x,\nu^{(k)})=\widetilde{\varphi}_k(x)$, $\widetilde{\beta}(x,\nu^{(k)})=\widetilde{\beta}_k(x)$, $\tau(x,\nu^{(k)})=\tau_k(x)$, $b_j(x,\nu^{(k)})=b_{jk}(x)$, $j=1,3$, $b_2(x,\nu^{(k)})=\rho_k(x)$, $\varepsilon^2(\nu^{(k)})=\varepsilon_k^2$.