L. ELSGOLTS

Differential Equations and the Calculus of Variations

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ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ И ВАРИАЦИОННОЕ ИСЧИСЛЕНИЕ

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The subject of this book is the theory of differential equations and the calculus of variations. It is based on a course of lectures which the author delivered for a number of years at the Physics Department of the Lomonosov State University of Moscow.

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PART ONE

Differential equations

Introduction

In the study of physical phenomena one is frequently unable to find directly the laws relating the quantities that characterize a phenomenon, whereas a relationship between the quantities and their derivatives or differentials can readily be established. One then obtains equations containing the unknown functions or vector functions under the sign of the derivative or differential.

Equations in which the unknown function or the vector function appears under the sign of the derivative or the differential are called *differential equations*. The following are some examples of differential equations:

(1) $\frac{dx}{dt} = -kx$ is the equation of radioactive disintegration (k is the disintegration constant, x is the quantity of undisintegrated substance at time t, and $\frac{dx}{dt}$ is the rate of decay proportional to the quantity of disintegrating substance).

(2) $m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}\left(t, \mathbf{r}, \frac{d\mathbf{r}}{dt}\right)$ is the equation of motion of a particle of mass m under the influence of a force $\mathbf{F}$ dependent on the time, the position of the particle (which is determined by the radius vector $\mathbf{r}$), and its velocity $\frac{d\mathbf{r}}{dt}$. The force is equal to the product of the mass by the acceleration.

(3) $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 4\pi\rho(x, y, z)$ is Poisson's equation, which for example is satisfied by the potential $u(x, y, z)$ of an electrostatic field, $\rho(x, y, z)$ is the charge density.

The relation between the sought-for quantities will be found if methods are indicated for finding the unknown functions which are defined by differential equations. The finding of unknown functions defined by differential equations is the principal task of the theory of differential equations.

If in a differential equation the unknown functions or the vector functions are functions of one variable, then the differential equation is called *ordinary* (for example, Eqs. 1 and 2 above). But if the unknown function appearing in the differential equation is a function of two or more independent variables, the differential equation is called a *partial differential equation* (Eq. 3 is an instance).

The *order* of a differential equation is the highest order of the derivative (or differential) of the unknown function.

A *solution* of a differential equation is a function which, when substituted into the differential equation, reduces it to an identity.

To illustrate, the equation of radioactive disintegration

$$\frac{dx}{dt} = -kx \tag{I.1}$$

has the solution

$$x = ce^{-kt}, \tag{I.1$_1$}$$

where c is an arbitrary constant.

It is obvious that the differential equation (I.1) does not yet fully determine the law of disintegration $x = x(t)$. For a full determination, one must know the quantity of disintegrating substance x_0 at some initial instant of time t_0. If x_0 is known, then, taking into account the condition $x(t_0) = x_0$ from (I.1$_1$), we find the law of radioactive disintegration:

$$x = x_0 e^{-k(t-t_0)}.$$

The procedure of finding the solutions of a differential equation is called *integration of the differential equation*. In the above case, it was easy to find an exact solution, but in more complicated cases it is very often necessary to apply approximate methods of integrating differential equations. Just recently these approximate methods still led to arduous calculations. Today, however, high-speed computers are able to accomplish such work at the rate of several hundreds of thousands of operations per second.

Let us now investigate more closely the above-mentioned more complicated problem of finding the law of motion $\mathbf{r} = \mathbf{r}(t)$ of a particle of mass m under the action of a specified force $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$. By Newton's law,

$$m\ddot{\mathbf{r}} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}}). \tag{I.2}$$

Consequently, the problem reduces to integrating this differential equation. Quite obviously, the law of motion is not yet fully defined by specifying the mass m and the force $\mathbf{F}$; one has also to know the initial position of the particle

$$\mathbf{r}(t_0) = \mathbf{r}_0 \tag{I.2$_1$}$$

and the initial velocity

$$\dot{\mathbf{r}}(t_0) = \dot{\mathbf{r}}_0. \tag{I.2$_2$}$$

We shall indicate an extremely natural approximate method for solving equation (I.2) with initial conditions (I.2$_1$) and (I.2$_2$); the idea of this method can also serve to prove the existence of a solution of the problem at hand.

We take the interval of time $t_0 \leqslant t \leqslant T$ over which it is required to find a solution of the equation (I.2) that will satisfy the initial conditions (I.2_1) and (I.2_2) and divide it into n equal parts of length $h = \frac{T - t_0}{n}$:

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, T],$$

where

$$t_k = t_0 + kh \qquad (k = 1, 2, \ldots, n-1).$$

For large values of n, within the limits of each one of these small intervals of time, the force $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$ changes but slightly (the vector function $\mathbf{F}$ is assumed to be continuous); therefore it may be taken, approximately, to be constant over every subinterval $[t_{k-1}, t_k]$, for instance, equal to the value it has at the left-hand boundary point of each subinterval. More exactly, on the subinterval $[t_0, t_1]$ the force $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$ is considered constant and equal to $\mathbf{F}(t_0, \mathbf{r}_0, \dot{\mathbf{r}}_0)$. On this assumption, it is easy, from (I.2) and the initial conditions (I.2_1) and (I.2_2), to determine the law of motion $\mathbf{r}_n(t)$ on the subinterval $[t_0, t_1]$ (the motion will be uniformly variable) and, hence, in particular, one knows the values of $\mathbf{r}_n(t_1)$ and $\dot{\mathbf{r}}_n(t_1)$. By the same method, we approximate the law of motion $\mathbf{r}_n(t)$ on the subinterval $[t_1, t_2]$ considering the force $\mathbf{F}$ as constant on this subinterval and as equal to $\mathbf{F}(t_1, \mathbf{r}_n(t_1), \dot{\mathbf{r}}_n(t_1))$. Continuing this process, we get an approximate solution $\mathbf{r}_n(t)$ to the posed problem with initial conditions for equation (I.2) over the whole interval $[t_0, T]$.

It is intuitively clear that as n tends to infinity, the approximate solution $\mathbf{r}_n(t)$ should approach the exact solution.

Note that the second-order vector equation (I.2) may be replaced by an equivalent system of two first-order vector equations if we regard the velocity $\mathbf{v}$ as the second unknown vector function:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = \mathbf{F}(t, \mathbf{r}, \mathbf{v}). \tag{I.3}$$

Every vector equation in three-dimensional space may be replaced by three scalar equations by projecting onto the coordinate axes. Thus, equation (I.2) is equivalent to a system of three scalar equations of the second order, and system (I.3) is equivalent to a system of six scalar equations of the first order.

Finally, it is possible to replace one second-order vector equation (I.2) in three-dimensional space by one vector equation of the first order in six-dimensional space, the coordinates here being r_x, r_y, r_z

of the radius vector $\mathbf{r}(t)$ and v_x, v_y, v_z of the velocity vector $\mathbf{v}$. *Phase space* is the term physicists use for this space. The radius vector $\mathbf{R}(t)$ in this space has the coordinates $(r_x, r_y, r_z, v_x, v_y, v_z)$. In this notation, (1.3) has the form

$$\frac{d\mathbf{R}}{dt} = \mathbf{\Phi}(t, \mathbf{R}(t)) \tag{1.4}$$

(the projections of the vector $\mathbf{\Phi}$ in six-dimensional space are the corresponding projections of the right-hand sides of the system (1.3) in three-dimensional space)

With this interpretation, the initial conditions (1.2_1) and (1.2_2) are replaced by the condition

$$\mathbf{R}(t_0) = \mathbf{R}_0 \tag{1.4_1}$$

The solution of (1.4) $\mathbf{R} = \mathbf{R}(t)$ will then be a phase trajectory, to each point of which there will correspond a certain instantaneous state of the moving particle—its position $\mathbf{r}(t)$ and its velocity $\mathbf{v}(t)$.

If we apply the above approximate method to (1.4) with initial condition (1.4_1), then on the first subinterval $[t_0, t_1]$ we must regard the vector function $\mathbf{\Phi}(t, \mathbf{R}(t))$ as constant and equal to $\mathbf{\Phi}(t_0, \mathbf{R}(t_0))$. And so, for $t_0 \leqslant t \leqslant t_0 + h$

$$\frac{d\mathbf{R}}{dt} = \mathbf{\Phi}(t_0, R(t_0));$$

from this, multiplying by dt and integrating between t_0 and t, we get the linear vector function $\mathbf{R}(t)$:

$$\mathbf{R}(t) = \mathbf{R}(t_0) + \mathbf{\Phi}(t_0, \mathbf{R}(t_0))(t - t_0).$$

In particular for $t = t_1$ we will have

$$\mathbf{R}(t_1) = \mathbf{R}(t_0) + h\mathbf{\Phi}(t_0, \mathbf{R}(t_0)).$$

Repeating the same reasoning for the subsequent subintervals, we get

$$\mathbf{R}(t_2) = \mathbf{R}(t_1) + h\mathbf{\Phi}(t_1, \mathbf{R}(t_1)),$$
$$\cdots\cdots\cdots\cdots$$
$$\mathbf{R}(t_k) = \mathbf{R}(t_{k-1}) + h\mathbf{\Phi}(t_{k-1}, \mathbf{R}(t_{k-1})),$$
$$\cdots\cdots\cdots\cdots$$

Applying these formulas n times we arrive at the value $\mathbf{R}(T)$.

In this method, the desired solution $\mathbf{R}(t)$ is approximately replaced by a piecewise linear vector function, the graph of which is a certain polygonal line called *Euler's polygonal curve.*

In applications, the problem for equation (1.2) is often posed differently: the supplementary conditions are specified at two points instead of one. Such a problem—unlike the problem with the