π Pitman Research Notes in Mathematics Series 253

M K V Murthy
and S Spagnolo (Editors)

Nonlinear hyperbolic equations and field theory

Longman
Scientific &
Technical

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Nonlinear hyperbolic equations and field theory

Longman Scientific & Technical
Copublished in the United States with
John Wiley & Sons, Inc., New York

Longman Scientific & Technical
Longman Group UK Limited
Longman House, Burnt Mill, Harlow
Essex CM20 2JE, England
and Associated Companies throughout the world.

Copublished in the United States with
John Wiley & Sons Inc., 605 Third Avenue, New York, NY 10158

First published 1992

AMS Classification: (Main) 35L, 53C, 81E, 81M
(Subsidiary) 58G, 83C, 76Y, 83D

ISSN 0269-3674

British Library Cataloguing in Publication Data

A catalogue record for this book is
available from the British Library

Library of Congress Cataloging-in-Publication Data

Nonlinear hyperbolic equations and field theory/M.K.V. Murthy &
S. Spagnolo (editors).
p. cm. -- (Pitman research notes in mathematics series, ISSN
0269-3674 ; 253)
Papers from a Workshop on Nonlinear Hyperbolic Equations, held in
Varenna, Italy, 1990.
Includes bibliographical references.
1. Differential equations, Hyperbolic--Congresses. 2. Field theory
(Physics)--Congresses. 3. Differential equations, Nonlinear-Congresses.
I. Murthy, M.K.V. (M.K. Venkatesha) II. Spagnolo, S. (Sergio) III.
Workshop on Nonlinear Hyperbolic Equations (1990 : Varenna, Italy)
IV. Series.
QA377 N665 1991
515'.353--dc20

91-18565
CIP

Printed and Bound in Great Britain
by Biddles Ltd, Guildford and King's Lynn

Pitman Research Notes in Mathematics Series

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S ALINHAC

Some remarks about the instability of the vortex patch problem

1. The vortex patch problem

1.1. Two dimensional flow.

We are considering incompressible flow in two dimensions in the whole of $\mathbb{R}^2$. If $v(x,t)$ is the velocity and $p(x,t)$ the pressure, the Euler equations are :

$$(1.1) \qquad \partial_t v + (v \cdot \nabla) v = -\nabla p \ , \quad \operatorname{div} v = 0 \ .$$

The vorticity $\omega = \partial_1 v_2 - \partial_2 v_1$ satisfies the equation

$$(1.2) \qquad \partial_t \omega + (v \cdot \nabla)\omega = 0 \ .$$

Here, v corresponds to ω through the formulas

$$(1.3) \qquad v = (-\partial_2 \psi, \partial_1 \psi) \ , \quad \Delta \psi = \omega \ .$$

A general framework to study such a flow has been given by YUDOVITCH [8] in the case of a bounded domain Ω with tangential speed at the boundary. A simplified version of this result is given by CHEMIN [3] in the case $\Omega = \mathbb{R}^2$ with ω small at infinity. Let $v_0 = v|_{t=0} = \sigma + w_0$, where σ is a rotation invariant stationnary solution of (1.1) and $w_0 \in L^2$, such that $w_0 \in L^\infty \cap L^q$ (for some $1 \le q < +\infty$). Then a (global in time) solution v exists to the Euler equations. It should be noted that only the L^p norms of ω are preserved with time (because of (1.2)), so that $|v(t,\cdot)|_{\text{lip}}$ need not be bounded. Nevertheless, $v(t,\cdot)$ is bounded in $C^1_*(\mathbb{R}^2)$, and the trajectories of the fluid particles (that is, the integral curves of v) exist and are unique (they are in fact C^∞, see CHEMIN [1]).

1.2. The vortex patch problem.

Suppose now that ω_0 is the characteristic function χ_{D_0} of a bounded domain D_0 (v_0 corresponding to ω_0 by (1.3)) : by (1.2), $\omega(t,\cdot)$ will be the characteristic function of another (bounded) domain D_t. Such a solution is determined by knowing D_t only : we have now a one dimensional problem. If the boundary of D_t is parametrized by $\tilde{z}(t,\alpha)$ ($\alpha \in \mathbb{R}$) and assumed to be smooth, then $\tilde{z}$ satisfies the equation

$$(1.4) \qquad \frac{\partial \tilde{z}}{\partial t}(t,\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \log |\tilde{z}(t,\alpha) - \tilde{z}(t,\beta)| \, \frac{\partial \tilde{z}}{\partial \beta}(t,\beta)\, d\beta \ .$$

A special case is obtained by taking the unit disk for D_0, the corresponding solution of (1.4) being then $\tilde{z}(t,\alpha) = e^{-i(t/2)}\, e^{i\alpha}$.

For smooth initial contours, CHEMIN [3] has proved local time existence of solutions of (1.4).

Our aim is to understand the large time behavior of solutions to (1.4).

Some numerical experiments [9] have shown a very irregular behavior of ∂D_t for finite time, leading MAJDA [6] to conjecture blow up of the length of ∂D_t.

In the following, we will provide some evidence of irregular behavior of ∂D_t, only in a special case that we will explain now.

1.3. The quadratic approximation.

a) The case where $\tilde{z}(0,\alpha)$ is taken close to $e^{i\alpha}$ is discussed in CONSTANTIN and TITI [4]. We set then $z(t,\alpha) = e^{i(t/2)}\tilde{z}(t,\alpha)$, and $z(t,\alpha) = e^{i\alpha} + \varphi(t,\alpha)$; the equation for φ is now

$$(1.5) \quad \frac{\partial\varphi}{\partial t}(t,\alpha) = \frac{1}{2}(i+H)\varphi + \\ + \frac{1}{2\pi}\int_0^{2\pi} \log\left|1 + \frac{\varphi(t,\alpha)-\varphi(t,\beta)}{e^{i\alpha}-e^{i\beta}}\right| \frac{\partial\left(e^{i\beta}+\varphi(t,\beta)\right)}{\partial\beta}\, d\beta$$

where H is the Hilbert operator, given by

$$H\left(\sum \varphi_k e^{ik\alpha}\right) = i\sum_{-\infty}^{-1} \varphi_k e^{ik\alpha} - i\sum_{1}^{+\infty} \varphi_k e^{ik\alpha} .$$

In this framework, SERFATI [7] proved local in time existence for $\varphi(0,\alpha) \in B_1$ and φ small enough. Here

$$B_1 = \left\{\varphi(\alpha) = \sum a_k e^{ik\alpha} ,\ \sum |k|\,|a_k| < +\infty\right\} .$$

b) If one retains only the linear terms in (1.5), one obtains (with $\varphi(t,\alpha) = \sum \varphi_k(t) e^{ik\alpha}$)

$$(1.6) \quad \begin{cases} \dfrac{d}{dt}\varphi_j = 0 , & j \geq 1 \\ \dfrac{d}{dt}\varphi_0 = \dfrac{i}{2}(\varphi_0 + \overline{\varphi}_2) & \\ \dfrac{d}{dt}\varphi_j = \dfrac{i}{2}(\varphi_j + \overline{\varphi}_{2-j}) , & j \leq -1 . \end{cases}$$

The solution is then easily computed and, for example, globally bounded in B_1 if $\varphi(0,\alpha) \in B_1$.

So no irregular behavior can be expected for the linearized problem.

c) If one retains linear **and quadratic terms** in (1.5), we obtain :

$$(1.7) \quad \frac{\partial\varphi}{\partial t} = \frac{1}{2}(i+H)\varphi +$$
$$+\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\frac{\varphi(t,\alpha)-\varphi(t,\beta)}{e^{i\alpha}-e^{i\beta}}\,\frac{\partial}{\partial\beta}\left(e^{i\beta}+\varphi(t,\beta)\right)d\beta$$
$$=\frac{1}{4\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{\varphi(t,\alpha)-\varphi(t,\beta)}{e^{i\alpha}-e^{i\beta}}\right)^2\frac{\partial e^{i\beta}}{\partial\beta}\,d\beta \equiv Q\varphi\ .$$

This is the simplest model for which one may expect the irregular behavior mentioned above.

We do not claim that one can deduce from the study of the quadratic approximation of (1.7) the behavior of the solutions of the full equation (1.5), though it is likely there is some qualitative similarity.

2. A "numerical blowup" theorem

THEOREM (see [**1**]). *For all* $\eta > 0$, $k \in \mathbf{N}$, $C > 0$, *there exists* T, $0 < T \leq 5$, *and a function* $\varphi(t,\alpha)$, C^∞ *on* $[0,T]\times\mathbf{R}$, *s.t. :*

i) *The function* φ *is an approximate solution to* (1.7) *:*

$$\frac{\partial\varphi}{\partial t} = Q\varphi + R\ , \qquad \sup_{t\in[0,T]} |R(t,\cdot)|_{C^k} \leq \eta\ .$$

ii)

$$\sup\ |\varphi| \leq \eta\ .$$

iii) *The initial contour* $z(0,\alpha) = e^{i\alpha} + \varphi(0,\alpha)$ *satisfies*

$$\sup_\alpha |z'_\alpha(0,\alpha)| \leq 14\ .$$

iv) *At time* T,

$$\sup_\alpha |z'_\alpha(T,\alpha)| \geq C\ .$$

In fact, one can give a slightly more sophisticated statement, replacing iii) *by*

iii)'

$$|z(0,\cdot)|_{B^1} \leq 14$$

and iv) *by*

iv)'

$$|z(T,\cdot)|_{C^1_*} \geq C\ .$$

This shows, when comparing with § 1.3.b), that the "numerical blowup" of the theorem is not due to an akward choice of norms, but to a quadratic effect.

The proof of the theorem relies on the construction of a family of asymptotic solutions to (1.7); in fact, the theorem summarizes only some information we can get from this family.

3. Rewriting "geometrically" the equation

One can write (1.7) as an infinite system on the Fourier coefficients of φ, similar to (1.6), but involving of course quadratic terms. However, it seems to be very hard to get from these equations the intuition of interesting solutions. To do this, one has to put emphasis on the two possible directions in the wave front of φ (in α) and their non linear antipodal interactions.

More precisely, we will look for φ in the form $\varphi(t,\alpha) = F(t,e^{i\alpha}) + G(t,e^{-i\alpha})$, where $F(t,z)$ and $G(t,z)$ are holomorphic near $\overline{D}_0 = \{|z| \leq 1\}$ (normalized with $F(t,0) = 0$). So, F corresponds to "$\xi = +1$" in $WF\varphi$, G to "$\xi = -1$".

To describe the interactions, let us introduce the following definition.

DEFINITION. For $u(z)$, $v(z)$ holomorphic near $\overline{D}_0$, let

$$h(u,v)(z) = \frac{1}{2i\pi} \int_{|z'|=1} \frac{u(z')v(\bar{z}')\,dz'}{z'-z} \,.$$

This is the "holomorphic part" of the product $u(z)v(\bar{z})$.

Note that if $u = $ cte , $h(u,v)$ is only a constant, while $h(u,v) = vu$ if σ is a constant.

In fact, v is "smoothed out" in $h(u,v)$: one can control a high norm of h by a high norms of u and a small norm of v (see [4]). In this sense, h behave qualitatively like a paraproduct $T_v u$ (see BONY [2]).

With this definition, equation (1.7) is equivalent to the following system :

$$(3.1)\qquad \begin{cases} \dfrac{\partial F}{\partial t}(t,z) = \dfrac{i}{2} S_0 h(\overline{G}, G' + \overline{H}')(t,z) \\ \dfrac{\partial G}{\partial t}(t,z) = \dfrac{i}{2}(G+\overline{H})(t,z) + \dfrac{i}{2}\Big\{ h(\overline{H}, F') - \\ \qquad\qquad - \overline{K} + S_2 h(\overline{H}, \overline{G}')\Big\}(t,z) \,. \end{cases}$$

Here $S_0 u(z) = u(z) - u(0)$, $S_2 u(z) = \dfrac{u(z) - u(0) - z u'(0)}{z^2}$, $H = S_2 F$, $K = HF/z$.

One can see that antipodal interactions appear as $h-$products, except for the term K.

4. Singular solutions to the quadratic equation

In order to obtain a solution $\varphi(t,\alpha)$ of (1.7) which is "as bad as possible", one tries to construct solutions F, G to (3.1) with sintularities very close to the unit circle.

If (3.1) were a quasi linear system, no solutions with poles like $\dfrac{1}{(z-p)^\ell}$ ($|p|>1$, $|p|$ close to one) could exist, because the order of the pole in the right hand side would be much bigger than ℓ. It is known in this case that ramified solutions can be constructed (see, for instance, LEICHTNAM [5]).

The crucial observation here is that the derivatives F', G' appear in (3.1) only as second factors in $h-$products : because of the smoothing effect discussed after the definition of h, singular solutions with poles will exist, as we now explain.

To see this, we use the following proposition.

PROPOSITION (see [**1**]). *For* p, $|p|>1$, $\nu = 1-|p|^2$ *and* $j,k \geq 1$, *we have*

$$h\left(\frac{1}{(z-p)^j}, \frac{1}{(z-\bar{p})^k}\right) = \sum_{\ell=0}^{j-1} C_{k+j-\ell-2}^{k-1} \frac{p^{k+\ell-j+1}}{(z-p)^{\ell+1}} \nu^{\ell-k-j+1}(1+O(\nu)),$$

where $O(\nu)$ *stands for a polynomial in* ν *(with coefficients independant of* z *and* p*) vanishing for* $\nu=0$.

If one tries for F and G the functions

$$F(t,z) = \frac{a(t)}{z-p}, \quad G(t,z) = \frac{b(t)}{z-\bar{p}} \quad (\text{for some } p,\ |p|>1)$$

the right hand-sides of (3.1) are again of the same form, except for the term K which behaves like $\dfrac{1}{(z-p)^2}$.

To take care of this, we have to introduce a concept of "ordering", necessary to define asymptotic solutions.

5. Ordering the terms by homogeneity

We shall use the following definition.

DEFINITION. A function $u(z)$ of the form

$$u(z) = \frac{\nu^k}{(z-p)^\ell} \quad (\nu = 1-|p|^2 \ ,\ |p|>1)$$

will be said homogeneous of degree $k-\ell$.

For such a function,

$$\sup\left|u(e^{i\alpha})\right| \sim |\nu|^{k-\ell} \ , \quad \sup\left|\partial_\alpha(u(e^{i\alpha}))\right| \sim |\nu|^{k-\ell-1} \qquad \text{etc} \ldots$$

Of course, if u is of degree d and v of degree d', then uv is of degree $d+d'$. The remarkable fact is that this holds also for $h-$products, as can be easily seen from the proposition in §. 3.

If now one tries for the main terms in F and G the functions

$$F^0(t,z) = \frac{a(t)\,\nu^2 p^2}{z-p} \;, \quad G^0(t,z) = \frac{b(t)\,\nu^2}{z-\bar{p}} \;, \tag{5.1}$$

homogeneous of degree 1, all the terms in the right hand sides of (3.1) are again of degree 1 with a simple pole, except for K which is of degree 2. Thus (3.1) can be satisfied up to remainder terms of degree 2 by such F^0, G^0, provided. a and b satisfy the system :

$$\begin{cases} \dfrac{da}{dt} = -\dfrac{i}{2}\,\bar{b}(b+\bar{a}) \;, \\ \dfrac{db}{dt} = \dfrac{i}{2}(b+\bar{a}) - \dfrac{1}{2}\,|a|^2 - \dfrac{i}{2}\,\bar{a}\bar{b} \;. \end{cases} \tag{5.2}$$

Note that this system is independant of p : this is the reason for the strange normalization of the therms in (5.1).

The next terms in F and G will be homogeneous of degree 2 with second ordre poles, that is of the form :

$$\begin{cases} F^1(t,z) = \dfrac{c(t)\,p^2}{z-p}\,\nu^3 + \dfrac{d(t)\,p^3}{(z-p)^2}\,\nu^4 \\ G^1(t,z) = \dfrac{e(t)}{z-\bar{p}}\,\nu^3 + \dfrac{f(t)\,\bar{p}}{(z-\bar{p})^2}\,\nu^4 \;. \end{cases} \tag{5.3}$$

Again, $F^0 + F^1$ and $G^0 + G^1$ satisfy (3.1) up to homogeneous terms of degree 3 provided c, d, e, f satisfy a linear system, depending on a and b, but not on p.

We can continue in the same way for higher order approximations (see [**1**] for details).

6. Blowing up the main terms

The irregular behavior of these solutions can be seen if one takes for a, b a solution to (5.2) which blows up in finite time :

PROPOSITION. *For all* $\varepsilon > 0$, $0 < T_\varepsilon < \dfrac{2}{\varepsilon}$ *and a solution* $a(t), b(t)$ *for* $t \in [0, T_\varepsilon[$ *to* (5.2) *of the form* $a(t) = 1 + b(t)$, $b(t) = -\dfrac{1}{2}(1+i)e(t)$, e *real,* $e(0) = 1+\varepsilon$, $e(t) \uparrow +\infty$ *when* $t \to T_\varepsilon$.

Once we have this, we take for instance the solution corresponding to $\varepsilon = \sqrt{2}-1$, $T_\varepsilon < \dfrac{2}{\sqrt{2}-1} \leq 5$. For this choice of a and b ; we choose T such that $\dfrac{7}{2}\,|b(T)| \geq 5 + 2C$, which insures point iv) of the theorem independantly of p.

After that, we can choose $|\nu|$ as small as we want to make φ small and all the remainder terms negligible.