Lectures on

PARTIAL DIFFERENTIAL EQUATIONS

By

G. B. FOLLAND

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
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PARTIAL DIFFERENTIAL EQUATIONS

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Notes by

K. T. JOSEPH and S. THANGAVELU

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Author

G. B. FOLLAND
University of Washington
Seattle, Washington 98175
U.S.A.

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PREFACE

This book consists of the notes for a course I gave at the T.I.F.R. Centre in Bangalore from September 20 to November 20, 1981. The purpose of the course was to introduce the students in the Programme in Applications of Mathematics to the applications of Fourier analysis - by which I mean the study of convolution operators as well as the Fourier transform itself - to partial differential equations. Faced with the problem of covering a reasonably broad spectrum of material in such a short time, I had to be selective in the choice of topics. I could not develop any one subject in a really thorough manner; rather, my aim was to present the essential features of some techniques that are well worth knowing and to derive a few interesting results which are illustrative of these techniques. This does not mean that I have dealt only with general machinery; indeed, the emphasis in Chapter 2 is on very concrete calculations with distributions and Fourier transforms - because the methods of performing such calculations are also well worth knowing.

If these notes suffer from the defect of incompleteness, they possess the corresponding virtue of brevity. They may therefore be of value to the reader who wishes to be introduced to some useful ideas without having to plough through a systematic treatise. More detailed accounts of the subjects discussed here can be found in the books of Folland [1], Stein[2], Taylor [3], and Trèves [4].

No specific knowledge of partial differential equations or Fourier Analysis is presupposed in these notes, although some prior acquaintance with the former is desirable. The main prerequisite is a familiarity with the subjects usually gathered under the rubric "real analysis" : measure and integration, and the elements of point set topology and functional analysis. In addition, the reader is expected to be acquainted with the basic facts about distributions as presented, for example, in Rudin [7].

I wish to express my gratitude to Professor K.G. Ramanathan for inviting me to Bangalore, and to Professor S. Raghavan and the staff of the T.I.F.R. Centre for making my visit there a most enjoyable one. I also wish to thank Mr S. Thangavelu and Mr K.T. Joseph for their painstaking job of writing up the notes.

CONTENTS

CHAPTER 1

PRELIMINARIES

IN THIS CHAPTER, we will study some basic results about convolutions and the Fourier transform.

§1. GENERAL THEOREMS ABOUT CONVOLUTIONS

We will begin with a theorem about integral operators.

THEOREM 1.1 Let K be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ such that, for some $c > 0$,

$$\int |K(x,y)|dy \leq c, \int |K(x,y)|dx \leq c, \text{ for every } x, y \text{ in } \mathbb{R}^n .$$

If $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$, then the function Tf, defined by

$$Tf(x) = \int K(x,y) f(y)dy \text{ for almost every } x \text{ in } \mathbb{R}^n ,$$

belongs to $L^p(\mathbb{R}^n)$ and further,

$$||Tf||_p \leq c ||f||_p .$$

PROOF If $p = \infty$, the hypothesis $\int |K(x,y)|dx \leq c$ is superfluous and the conclusion of the theorem is obvious. If $p < \infty$, let q denote the conjugate exponent. Then, by Hölder's inequality,

$$\begin{aligned} |Tf(x)| &\leq \{\int |K(x,y)|dy\}^{1/q} \{\int |K(x,y)||f(y)|^p dy\}^{1/p} \\ &\leq c^{1/q} \{\int |K(x,y)||f(y)|^p dy\}^{1/p} . \end{aligned}$$

From this we have,

$$\begin{aligned} \int |Tf(x)|^p dx &\leq c^{p/q} \iint |K(x,y)||f(y)|^p dy\, dx \\ &\leq c^{1+p/q} \int |f(y)|^p dx = c^{1+p/q} ||f||_p^p . \end{aligned}$$

Therefore $\|Tf\|_p \leq c \|f\|_p$.

Next, we define the convolution of two locally integrable functions.

<u>DEFINITION 1.2</u> Let f and g be two locally integrable functions. The <u>convolution</u> of f and g, denoted by $f * g$, is defined by

$$(f*g)(x) = \int f(x-y)g(y)dy = \int f(y)g(x-y)dy = (g*f)(x) ,$$

provided that the integrals in question exist.

(The two integrals are equal by the change of variable $y \to x - y$.)

The basic theorem on convolutions is the following theorem, called <u>Young's inequality</u>.

<u>THEOREM 1.3</u> (<u>Young's Inequality</u>) <u>Let</u> $f \in L^1(\mathbb{R}^n)$ <u>and</u> $g \in L^p(\mathbb{R}^n)$, <u>for</u> $1 \leq p \leq \infty$. <u>Then</u> $f * g \in L^p(\mathbb{R}^n)$ <u>and</u>

$$\|f * g\|_p \leq \|g\|_p \|f\|_1 .$$

<u>PROOF</u> Take $K(x,y) = f(x - y)$ in Theorem 1.1. Then $K(x,y)$ satisfies all the conditions of Theorem 1.1 and the conclusion follows immediately.

The next theorem underlies one of the most important uses of convolution. Before coming to the theorem, let us prove the following

<u>LEMMA 1.4</u> <u>For a function</u> f <u>defined on</u> $\mathbb{R}^n$ <u>and</u> x <u>in</u> $\mathbb{R}^n$, <u>we define a function</u> f^x <u>by</u> $f^x(y) = f(y-x)$. <u>If</u> $f \in L^p$, $1 \leq p < \infty$, <u>then</u> $\lim_{x \to 0} \|f^x - f\|_p = 0$.

PROOF If g is a compactly supported continuous function, then g is uniformly continuous, and so g^x converges to g uniformly as x tends to 0.

Further, for $|x| \leq 1$, g^x and g are supported in a common compact set. Therefore, $\lim_{x \to 0} \|g^x - g\|_p = 0$. Given $f \in L^p$, we can find a function g which is continuous and compactly supported such that $\|f-g\|_p < \varepsilon/3$ for $\varepsilon > 0$. But then $\|g^x - f^x\|_p < \varepsilon/3$ also holds. Therefore

$$\|f^x - f\|_p \leq \|f^x - g^x\|_p + \|g^x - g\|_p + \|g-f\|_p$$
$$\leq 2\varepsilon/3 + \|g^x - g\|_p.$$

Since $\lim_{x \to 0} \|g^x - g\|_p = 0$, we can choose x close to 0 so that $\|g^x - g\|_p < \varepsilon/3$. Then $\|f^x - f\| < \varepsilon$ and this proves the lemma since ε is arbitrary.

REMARK 1.5 The above lemma is false when $p = \infty$. Indeed, "$f^x \to f$ in L^∞" means precisely that f is uniformly continuous.

Let us now make two important observations about convolutions which we shall use without comment later on.

i) $\text{Supp}(f * g) \subset \text{Supp } f + \text{Supp } g$, where

$$A+B = \{x+y : x \in A, y \in B\}.$$

ii) If f is of class C^k and $\partial^\alpha f (|\alpha| \leq k)$ and g satisfy appropriate conditions so that differentiation under the integral sign is justified, then $f * g$ is of class C^k and $\partial^\alpha(f * g) = (\partial^\alpha f) * g$.

THEOREM 1.6 *Let* $g \in L^1(\mathbb{R}^n)$ *and* $\int g(x)dx = a$. *Let* $g_\varepsilon(x) = \varepsilon^{-n} g(x/\varepsilon)$ *for* $\varepsilon > 0$. *Then, we have the following:*

i) *If* $f \in L^p(\mathbb{R}^n)$, $p < \infty$, $f*g_\varepsilon$ *converges to* af *in* L^p *as* ε *tends to* 0.

ii) *If* f *is bounded and continuous, then* $f*g_\varepsilon$ *converges to* af *uniformly on compact sets as* ε *tends to* 0.

PROOF By the change of variable $x \to \varepsilon x$, we see that $\int g_\varepsilon(x)dx = a$ for all $\varepsilon > 0$. Now

$$\begin{aligned}(f*g_\varepsilon)(x) - af(x) &= \int f(x-y)g_\varepsilon(y)dy - \int f(x)g_\varepsilon(y)dy \\ &= \int [f(x-y)-f(x)]\, g_\varepsilon(y)dy \\ &= \int [f(x-\varepsilon y) - f(x)]\, g(y)dy \\ &= \int [f^{\varepsilon y}(x) - f(x)]\, g(y)dy.\end{aligned}$$

If $f \in L^p$ and $p < \infty$, we apply Minkowski's inequality for integrals to obtain

$$\| f*g_\varepsilon - af \|_p \le \int \| f^{\varepsilon y} - f \|_p |g(y)| dy.$$

The function $y \to \| f^{\varepsilon y} - f \|_p$ is bounded by $2\|f\|_p$ and tends to 0 as ε tends to 0 for each y, by Lemma 1.4. Therefore, we can apply Lebesgue Dominated Convergence theorem to get the desired result.

On the other hand, suppose f is bounded and continuous. Let K be any compact subset of $\mathbb{R}^n$. Given $\delta > 0$, choose a compact set G $\mathbb{R}^n$ such that

Let us now consider the Fourier transform in the Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$.

<u>PROPOSITION 1.12</u> <u>For</u> $f \in \mathcal{S}$, <u>we have the following</u> :

i) $\hat{f} \in C^\infty(\mathbb{R}^n)$ <u>and</u> $\partial^\beta \hat{f} = \hat{g}$ <u>where</u> $g(x) = (-2\pi i x)^\beta f(x)$.

ii) $(\partial^\beta f)^\wedge(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi)$.

<u>PROOF</u> i) Differentiation under the integral sign proves this.

ii) For this, we use integration by parts.

$$\begin{aligned}(\partial^\beta f)^\wedge(\xi) &= \int e^{-2\pi i x\cdot\xi}(\partial^\beta f)(x)dx \\ &= (-1)^{|\beta|} \int \partial^\beta[e^{-2\pi i x\cdot\xi}]f(x)dx \\ &= (-1)^{|\beta|}(-2\pi i\xi)^\beta \int e^{-2\pi i x\cdot\xi} f(x)dx \\ &= (2\pi i\xi)^\beta \hat{f}(\xi).\end{aligned}$$

<u>COROLLARY 1.13</u> <u>If</u> $f \in \mathcal{S}$, <u>then</u> $\hat{f} \in \mathcal{S}$ <u>also</u>.

<u>PROOF</u> For multi-indices α and β, using Proposition 1.12, we have

$$\begin{aligned}\xi^\alpha(\partial^\beta \hat{f})(\xi) &= \xi^\alpha((-2\pi i x)^\beta f(x))^\wedge(\xi) \\ &= (2\pi i)^{-|\alpha|}[\partial^\alpha((-2\pi i x)^\beta f(x))^\wedge(\xi)] \\ &= (-1)^{|\beta|}(2\pi i)^{|\beta|-|\alpha|}(\partial^\alpha(x^\beta f(x)))^\wedge(\xi)\end{aligned}$$

Since $f \in \mathcal{S}, \partial^\alpha(x^\beta f(x)) \in L^1$ and hence $(\partial^\alpha(x^\beta f(x)))^\wedge \in L^\infty$. Thus $\xi^\alpha(\partial^\beta \hat{f})$ is bounded. Since α and β are arbitrary, this proves that $\hat{f} \in \mathcal{S}$.

<u>COROLLARY 1.14</u> <u>(RIEMANN-LEBESGUE LEMMA)</u> <u>If</u> $f \in L^1$, <u>then</u> $\hat{f}$ <u>is continuous and vanishes at</u> ∞.

PROOF Let $V = \{x \in \Omega : d(x,K) \le \frac{1}{2}\delta\}$ where $\delta = d(K, \mathbb{R}^n \setminus \Omega)$. Choose a $\phi_o \in C_o^\infty$ such that Supp $\phi_o \subset B(0, \frac{1}{2}\delta)$ and $\int \phi_o(x)dx = 1$. Define

$$\phi(x) = \int_V \phi_o(x-y)dy = (\phi_o * \chi_V)(x).$$

Then $\phi(x)$ is a function with the required properties.

§2. THE FOURIER TRANSFORM

In this section, we will give a rapid introduction to the theory of the Fourier transform.

For a function $f \in L^1(\mathbb{R}^n)$, the Fourier transform of the function f, denoted by $\hat{f}$, is defined by

$$\hat{f}(\xi) = \int e^{-2\pi i\, x\cdot\xi} f(x)dx, \ \xi \in \mathbb{R}^n.$$

REMARK 1.9 Our definition of $\hat{f}$ differs from some others in the placement of the factor of 2π.

BASIC PROPERTIES OF THE FOURIER TRANSFORM

(1.10) For $f \in L^1$, $\|\hat{f}\|_\infty \le \|f\|_1$.

The proof of this is trivial.

(1.11) For $f, g \in L^1$, $(f*g)^\wedge(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

Indeed,

$$\begin{aligned}(f*g)^\wedge(\xi) &= \iint e^{-2\pi i\, x\cdot\xi} f(y)\, g(x-y)dy\, dx \\ &= \iint e^{-2\pi i(x-y)\cdot\xi} g(x-y)\, e^{-2\pi i y\cdot\xi} f(y)\, dydx \\ &= \int e^{-2\pi i(x-y)\cdot\xi} g(x-y)\, dx \int f(y)\, e^{-2\pi i y\cdot\xi} dy \\ &= \hat{f}(\xi)\hat{g}(\xi).\end{aligned}$$

Let us now consider the Fourier transform in the Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$.

PROPOSITION 1.12 *For* $f \in \mathcal{S}$, *we have the following* :

i) $\hat{f} \in C^\infty(\mathbb{R}^n)$ *and* $\partial^\beta \hat{f} = \hat{g}$ *where* $g(x) = (-2\pi i x)^\beta f(x)$.

ii) $(\partial^\beta f)^\wedge(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi)$.

PROOF i) Differentiation under the integral sign proves this.

ii) For this, we use integration by parts.

$$\begin{aligned}
(\partial^\beta f)^\wedge(\xi) &= \int e^{-2\pi i x\cdot\xi}(\partial^\beta f)(x)dx \\
&= (-1)^{|\beta|} \int \partial^\beta[e^{-2\pi i x\cdot\xi}]f(x)dx \\
&= (-1)^{|\beta|} (-2\pi i\xi)^\beta \int e^{-2\pi i x\cdot\xi} f(x)dx \\
&= (2\pi i\xi)^\beta \hat{f}(\xi).
\end{aligned}$$

COROLLARY 1.13 *If* $f \in \mathcal{S}$, *then* $\hat{f} \in \mathcal{S}$ *also*.

PROOF For multi-indices α and β, using Proposition 1.12, we have

$$\begin{aligned}
\xi^\alpha(\partial^\beta \hat{f})(\xi) &= \xi^\alpha((-2\pi i x)^\beta f(x))^\wedge(\xi) \\
&= (2\pi i)^{-|\alpha|}[\partial^\alpha((-2\pi i x)^\beta f(x))^\wedge(\xi)] \\
&= (-1)^{|\beta|}(2\pi i)^{|\beta|-|\alpha|} (\partial^\alpha(x^\beta f(x)))^\wedge(\xi)
\end{aligned}$$

Since $f \in \mathcal{S}, \partial^\alpha(x^\beta f(x)) \in L^1$ and hence $(\partial^\alpha(x^\beta f(x)))^\wedge \in L^\infty$. Thus $\xi^\alpha(\partial^\beta \hat{f})$ is bounded. Since α and β are arbitrary, this proves that $\hat{f} \in \mathcal{S}$.

COROLLARY 1.14 (RIEMANN-LEBESGUE LEMMA) *If* $f \in L^1$, *then* $\hat{f}$ *is continuous and vanishes at* ∞.

PROOF By Corollary 1.13, this is true for $f \in S$. Since S is dense in L^1 and $\|\hat{f}\|_\infty \leq \|f\|_1$, the same is true for all $f \in L^1$.

Let us now compute the Fourier transform of the Gaussian.

THEOREM 1.15 Let $f(x) = e^{-\pi a|x|^2}$, Re $a > 0$. Then, $\hat{f}(\xi) = a^{-n/2} e^{-a^{-1}\pi|\xi|^2}$.

PROOF $\hat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} e^{-a\pi|x|^2} dx$

i.e. $$\hat{f}(\xi) = \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-2\pi i x_j \xi_j} e^{-a\pi x_j^2} dx_j .$$

Thus it suffices to consider the case $n = 1$. Further, we can take $a=1$ by making the change of variable $x \to a^{-1/2}x$.

Thus we are assuming $f(x) = e^{-\pi x^2}$, $x \in \mathbb{R}$. Observe that $f'(x) + 2\pi x f(x) = 0$. Taking the Fourier transform, we obtain

$$2\pi i\xi \hat{f}(\xi) + i \hat{f}'(\xi) = 0 .$$

Hence

$$\hat{f}'(\xi)/f(\xi) = -2\pi\xi$$

which, on integration, gives $f(\xi) = c\, e^{-\pi\xi^2}$, c being a constant.

The constant c is given by

$$c = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Therefore $\hat{f}(\xi) = e^{-\pi\xi^2}$, which completes the proof.

We now derive the Fourier inversion formula for the Schwartz class S.

Let us define $f^{\vee}(\xi) = \int e^{2\pi i x\cdot\xi} f(x)dx = \hat{f}(-\xi)$.

THEOREM 1.16 (Fourier Inversion Theorem) *For* $f \in \mathcal{S}, (\hat{f})^{\vee} = f$.

PROOF First, observe that for $f, g \in L^1$, $\int f\,\hat{g} = \int \hat{f}\,g$.

In fact,

$$\int \hat{f}(x)g(x)dx = \iint e^{-2\pi i y\cdot x} f(y)g(x)dy\,dx$$

$$= \int [\int e^{-2\pi i y\cdot x} g(x)dx]\, f(y)dy$$

$$= \int \hat{g}(y)\, f(y)dy.$$

Given $\varepsilon > 0$ and x in $\mathbb{R}^n$, take the function ϕ defined by

$$\phi(\xi) = e^{2\pi i x\cdot\xi - \pi\varepsilon^2|\xi|^2}.$$

Now

$$\hat{\phi}(y) = \int e^{-2\pi i y\cdot\xi}\, e^{2\pi i x\cdot\xi - \pi\varepsilon^2|\xi|^2} d\xi$$

$$= \int e^{-2\pi i (y-x)\cdot\xi} e^{-\pi\varepsilon^2|\xi|^2} d\xi$$

$$= \varepsilon^{-n}\, e^{-\pi\varepsilon^{-2}|x-y|^2}.$$

If we take $g(x) = e^{-\pi|x|^2}$ and define $g_\varepsilon(x) = \varepsilon^{-n}\, g(x/\varepsilon)$, then

$$\hat{\phi}(y) = g_\varepsilon(x-y).$$

Therefore

$$\int e^{2\pi i x\cdot\xi}\, \hat{f}(\xi)\, e^{-\pi\varepsilon^2|\xi|^2} d\xi$$

$$= \int \hat{f}(\xi)\, \phi(\xi)d\xi$$

$$= \int f(y)\, \hat{\phi}(y)dy$$

$$= \int f(y)\, g_\varepsilon(x-y)dy$$

$$= (f * g_\varepsilon)\,(x)$$

But as ε tends to 0, $(f * g_\varepsilon)$ converges to f, by Theorem 1.6 and clearly

$$\int e^{2\pi i x\cdot\xi}\hat{f}(\xi)e^{-\pi\varepsilon^2|\xi|^2}\,d\xi \to \int \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi \; .$$

Therefore $(\hat{f})^{\vee} = f$.

<u>COROLLARY 1.17</u> <u>The Fourier transform is an isomorphism of</u> $\mathcal{S}$ <u>onto</u> $\mathcal{S}$.

Next, we prove the <u>Plancherel Theorem</u>.

<u>THEOREM 1.18</u> <u>The Fourier transform uniquely extends to a unitary map of</u> $L^2(\mathbb{R}^n)$ <u>onto itself</u>.

<u>PROOF</u> For $f \in \mathcal{S}$, define $\tilde{f}(x) = \overline{f(-x)}$. Then it is easily checked that $\hat{\tilde{f}} = \bar{\hat{f}}$, so that

$$\begin{aligned}\|f\|_2^2 &= \int |f(x)|^2 dx\\ &= \int f(x)\;\tilde{f}(-x)dx\\ &= (f * \tilde{f})\;(0)\\ &= \int (f * \tilde{f})^{\wedge}(\xi)d\xi\\ &= \int \hat{f}(\xi)\;\hat{\tilde{f}}(\xi)d\xi\\ &= \int \hat{f}(\xi)\;\bar{\hat{f}}(\xi)d\xi = \|\hat{f}\|_2^2 \; .\end{aligned}$$

Therefore, the Fourier transform extends continuously to an isometry of L^2 . It is a unitary transformation, since its image $\mathcal{S}$ is dense in L^2 .

Let us observe how the Fourier transform interacts with translations, rotations and dilations.