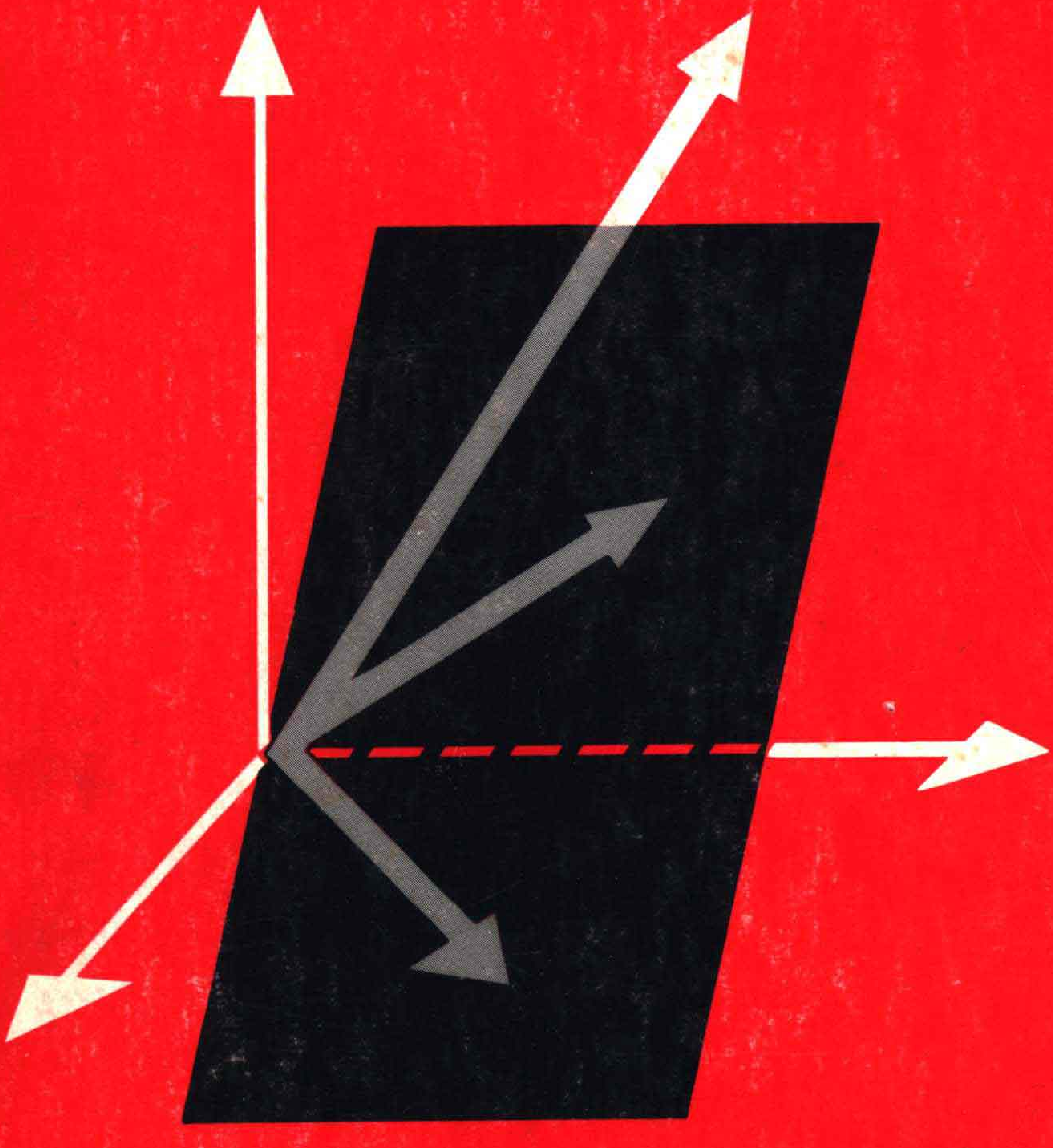
Chris Rorres
Howard Anton

Applications of Linear Algebra

SECOND EDITION



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Preface

This textbook discusses selected applications of linear algebra. The presentation is suitable for students who have completed or are taking concurrently a standard sophomore-level course in linear algebra.

Topics are drawn from a wide variety of fields including business, economics, engineering, physics, computer science, geometry, approximation theory, ecology, sociology, demography, and genetics. Also included is a brief introduction to game theory, Markov chains, and graph theory. At the end of the text there is a three chapter minicourse in linear programming which can be covered in about six lectures.

With a few clearly-marked exceptions, each application is in its own *independent* chapter, so that chapters can be deleted or permuted freely to fit individual needs and interests. Each topic begins with a list of linear algebra prerequisites in order that a reader can tell in advance if he or she has sufficient background to read the chapter.

Since the topics vary considerably in difficulty, we have included a *subjective* rating of each topic — easy, moderate, more difficult. (See the *Guide for the Instructor* following this preface.) Our evaluation is based more on the intrinsic difficulty of the material rather than the number of prerequisites; thus a topic requiring fewer mathematical prerequisites may be rated harder than one requiring more prerequisites.

Since our primary objective is to present applications of linear algebra, proofs are often omitted. We assume the reader has met the linear algebra prerequisites and whenever results from other fields are needed, they are stated precisely (with motivation where possible), but usually without proof.

Although this text was written to be used with Howard Anton's *Elementary Linear Algebra*, John Wiley and Sons, Inc., we have avoided specialized notation or terminology so that this book can be utilized in conjunction with any standard undergraduate text in linear algebra.

There are several possible ways to use this book:

(a) as a supplement to a standard linear algebra text,
(b) as a textbook for a follow-up course to linear algebra,
(c) as part of a self-study enrichment program or an introduction to mathematical research.

In addition this text may serve as a source of topics for a mathematical modeling or computer programming course.

Four new chapters are included in this second edition:

- Plane Geometry
- Equilibrium of Rigid Bodies
- The Assignment Problem
- Computer Graphics

A solutions manual has also been prepared in conjunction with the second edition. It contains complete solutions to all of the exercises in the text.

We would like to express our appreciation to Kathleen R. McCabe of Techni-Type who typed the entire manuscript. Her patience and skill contributed greatly to the appearance of this text. Our thanks are also due to Charles Shuman, who assisted with the exercises and examples, and Dennis DeTurck, who prepared the solutions manual. We also express our appreciation for the guidance provided by the reviewers: Keith J. Craswell of Western Washington University, Harry W. McLaughlin of Rensselaer Polytechnic Institute, and Roberto Mena of the University of Wyoming. Finally, we thank J. Robert Parker for his artistic assistance and the entire Wiley staff, especially Judy Hirsch and Gary Ostedt, for their encouragement and guidance.

Chris Rorres

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Contents

1

Constructing Curves and Surfaces through Specified Points

A technique for using determinants to construct lines, circles, and general conic sections through specified points in the plane is described. The procedure is also used to pass planes and spheres in three-dimensional space through fixed points.

PREREQUISITES: Linear systems
Determinants
Analytic geometry

INTRODUCTION

One of the fundamental results in the theory of Linear Algebra is the following:

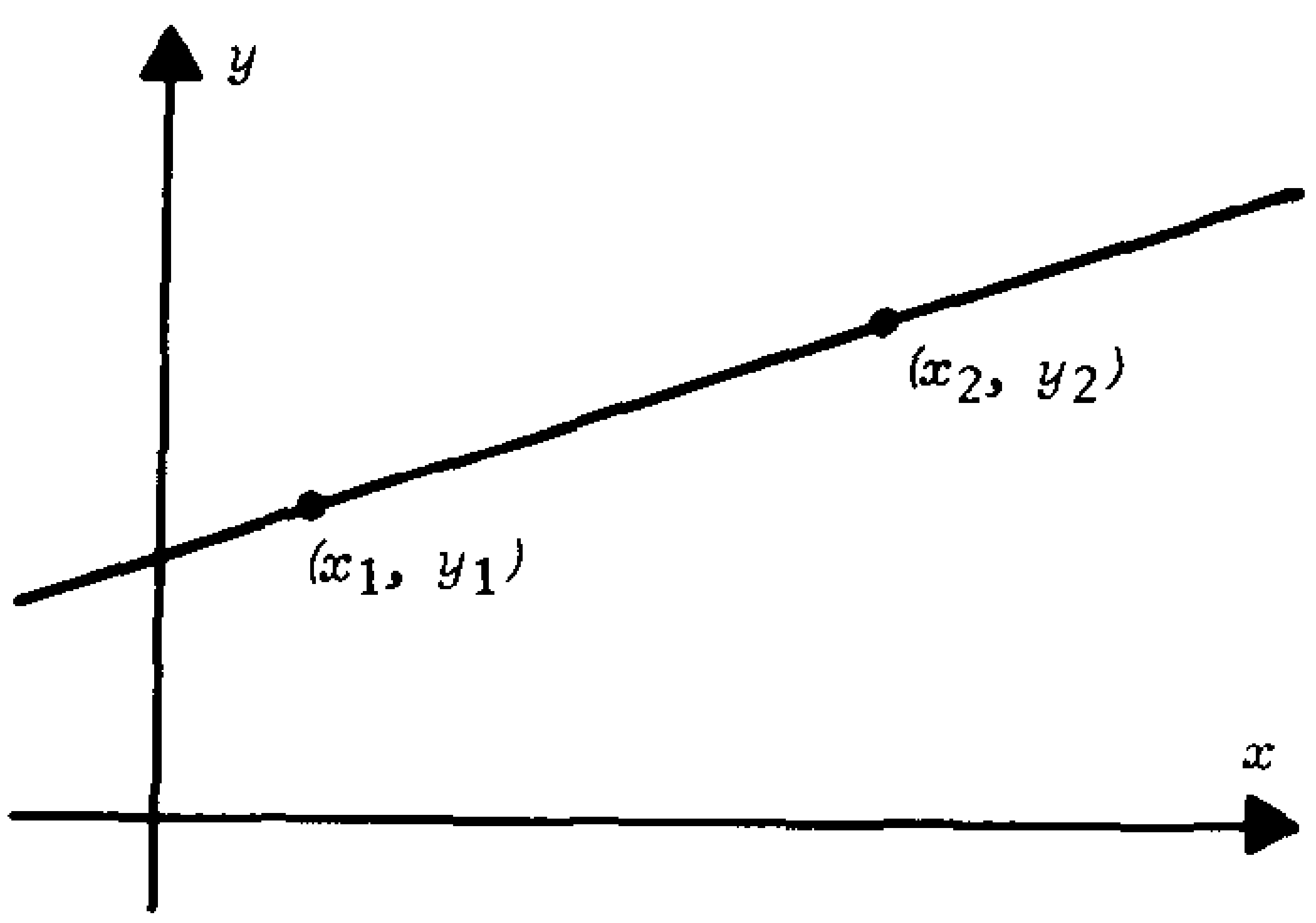
> *A homogeneous linear system with as many equations as unknowns has a nontrivial solution if and only if the determinant of the system is zero.*

In this chapter, we show how this result may be used to determine the equations of various curves and surfaces through specified points. We proceed immediately to some specific examples.

A Line Through Two Points

Suppose we are given two distinct points in the plane, (x_1, y_1) and (x_2, y_2). There is a unique line,

$$c_1x + c_2y + c_3 = 0, \qquad (1.1)$$



which passes through these two points. Notice that c_1, c_2, and c_3 are not all zero, and that these coefficients are unique only up to a multiplicative constant. Since (x_1, y_1) and (x_2, y_2) lie on the line, substituting them in (1.1) gives the two equations

$$c_1x_1 + c_2y_1 + c_3 = 0 \qquad (1.2)$$

$$c_1x_2 + c_2y_2 + c_3 = 0. \qquad (1.3)$$

The three equations, (1.1), (1.2), and (1.3), may be grouped together in system form as

$$\begin{aligned} x\,c_1 + y\,c_2 + c_3 &= 0 \\ x_1c_1 + y_1c_2 + c_3 &= 0 \\ x_2c_1 + y_2c_2 + c_3 &= 0. \end{aligned}$$

In this form, we have a homogeneous system of three equations for c_1, c_2, and c_3. Since c_1, c_2, and c_3 are not all zero, this system has a nontrivial solution, and so the determinant of the system must be zero. That is,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \qquad (1.4)$$

Consequently, every point (x, y) on the line satisfies (1.4), and conversely, every point (x, y) which satisfies (1.4) lies on the line.

Example 1.1 Find the equation of the line which passes through the two points (2, 1) and (3, 7).

Solution Substitution of the coordinates of the two points into Eq. (1.4) gives

$$\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 3 & 7 & 1 \end{vmatrix} = 0.$$

The cofactor expansion of this determinant along the first row then gives:

$$-6x + y + 11 = 0.$$

A Circle Through Three Points

Let us be given three distinct points in the plane, (x_1, y_1), (x_2, y_2), and (x_3, y_3), not all lying on a straight line. From analytic geometry, we know that there is a unique circle, say

$$c_1(x^2 + y^2) + c_2x + c_3y + c_4 = 0, \quad (1.5)$$

which passes through them (Fig. 1.2). Substituting the coordinates of the three points into this equation gives

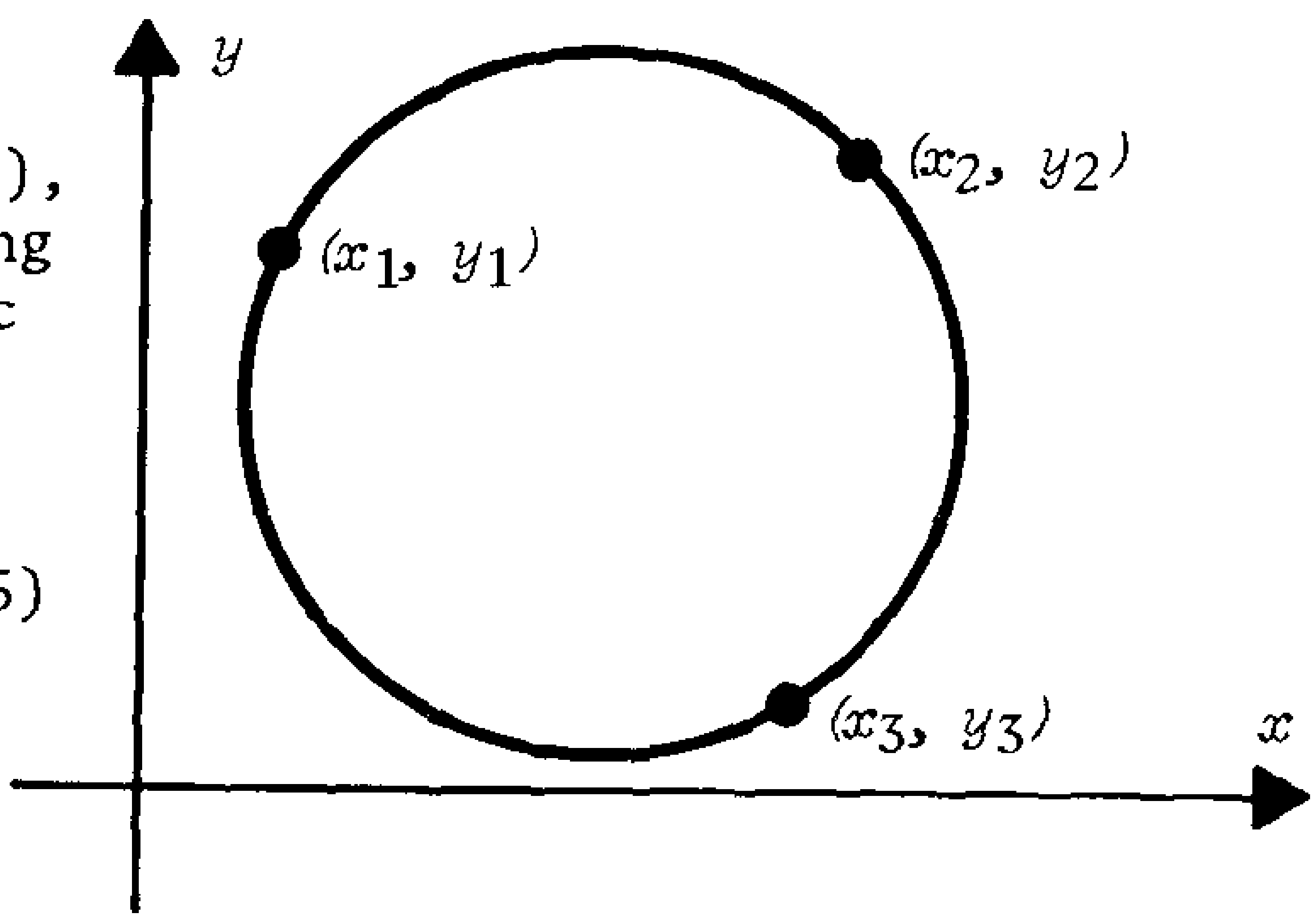


Figure 1.2

$$c_1(x_1^2 + y_1^2) + c_2x_1 + c_3y_1 + c_4 = 0 \quad (1.6)$$

$$c_1(x_2^2 + y_2^2) + c_2x_2 + c_3y_2 + c_4 = 0 \quad (1.7)$$

$$c_1(x_3^2 + y_3^2) + c_2x_3 + c_3y_3 + c_4 = 0. \quad (1.8)$$

As before, Eqs. (1.5) - (1.8) form a homogeneous linear system with a nontrivial solution for c_1, c_2, c_3, and c_4. Thus the determinant of this linear system is zero:

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_2 & y_3 & 1 \end{vmatrix} = 0. \qquad (1.9)$$

This is a determinant form for the equation of the circle.

Example 1.2 Find the equation of the circle which passes through the three points (1, 7), (6, 2), and (4, 6).

Solution Substitution of the coordinates of the three points into Eq. (1.9) gives

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 50 & 1 & 7 & 1 \\ 40 & 6 & 2 & 1 \\ 52 & 4 & 6 & 1 \end{vmatrix} = 0,$$

which reduces to

$$10(x^2+y^2) - 20x - 40y - 200 = 0.$$

In standard form this is

$$(x-1)^2 + (y-2)^2 = 5^2.$$

Thus, the circle has center (1, 2) and radius 5.

A General Conic Section Through Five Points

The general equation of a conic section in the plane (a parabola, hyperbola, or ellipse, and degenerate forms of these three curves) is given by

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0.$$

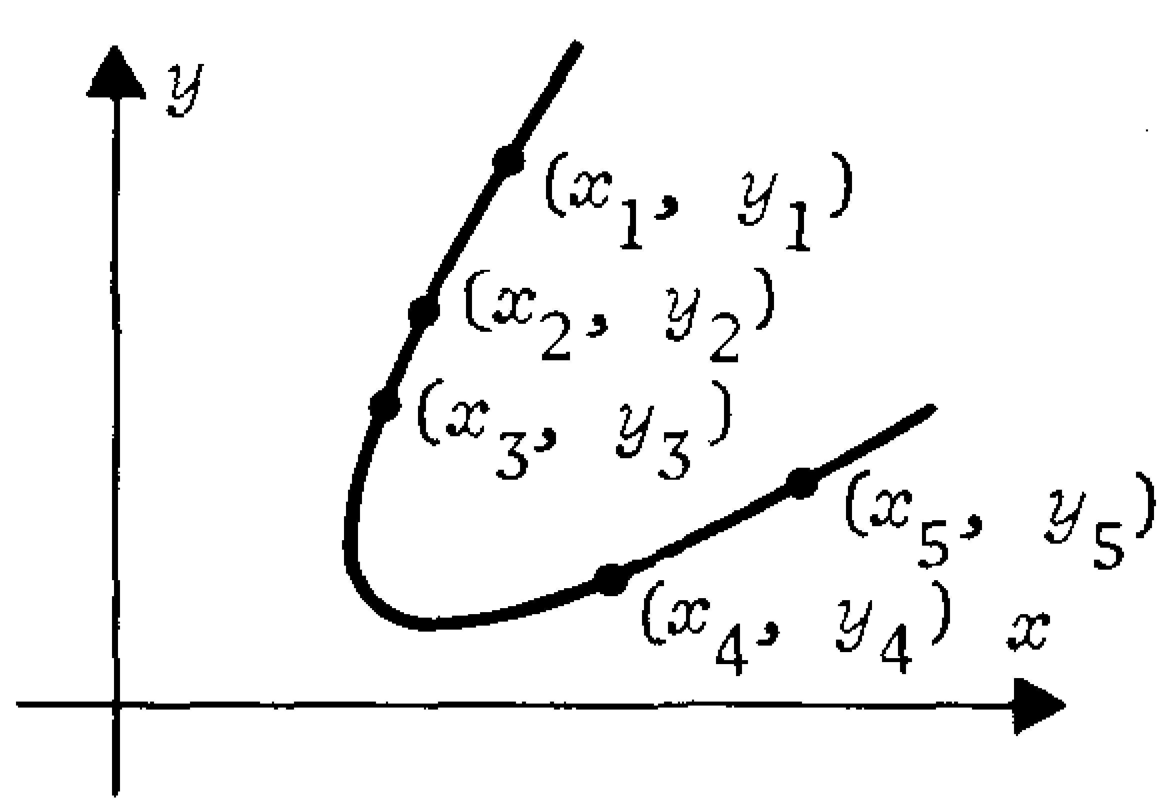


Figure 1.3

This form contains six coefficients, although only five are needed since we may divide through by any one of them which is not zero. Thus, only five coefficients must be determined, so that five distinct points in the plane are sufficient to determine the equation of the conic section (Fig. 1.3). As before, the equation may be put in determinant form (see Exercise 1.6):

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0. \tag{1.10}$$

EXAMPLE 1.3 An astronomer wants to determine the orbit of an asteroid about the sun. He sets up a Cartesian coordinate system in the plane of the orbit with the sun at the origin. Astronomical units of measurement are used along the axes. (1 astronomical unit = mean distance of earth to sun = 93 million miles.) By Kepler's first law, he knows that the orbit must be an ellipse. Consequently, he makes five observations of the asteroid at five different times and finds five points along the orbit to be

(5.764,0.648), (6.286,1.202), (6.759,1.823),
(7.168,2.526), (7.480,3.360).

Find the equation of the orbit.

SOLUTION Substitution of the coordinates of the five given points into Eq. (1.10) gives

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 33.224 & 3.735 & 0.420 & 5.764 & 0.648 & 1 \\ 39.514 & 7.556 & 1.445 & 6.286 & 1.202 & 1 \\ 45.684 & 12.322 & 3.323 & 6.759 & 1.823 & 1 \\ 51.380 & 18.106 & 6.381 & 7.168 & 2.526 & 1 \\ 55.950 & 25.133 & 11.290 & 7.480 & 3.360 & 1 \end{vmatrix} = 0.$$

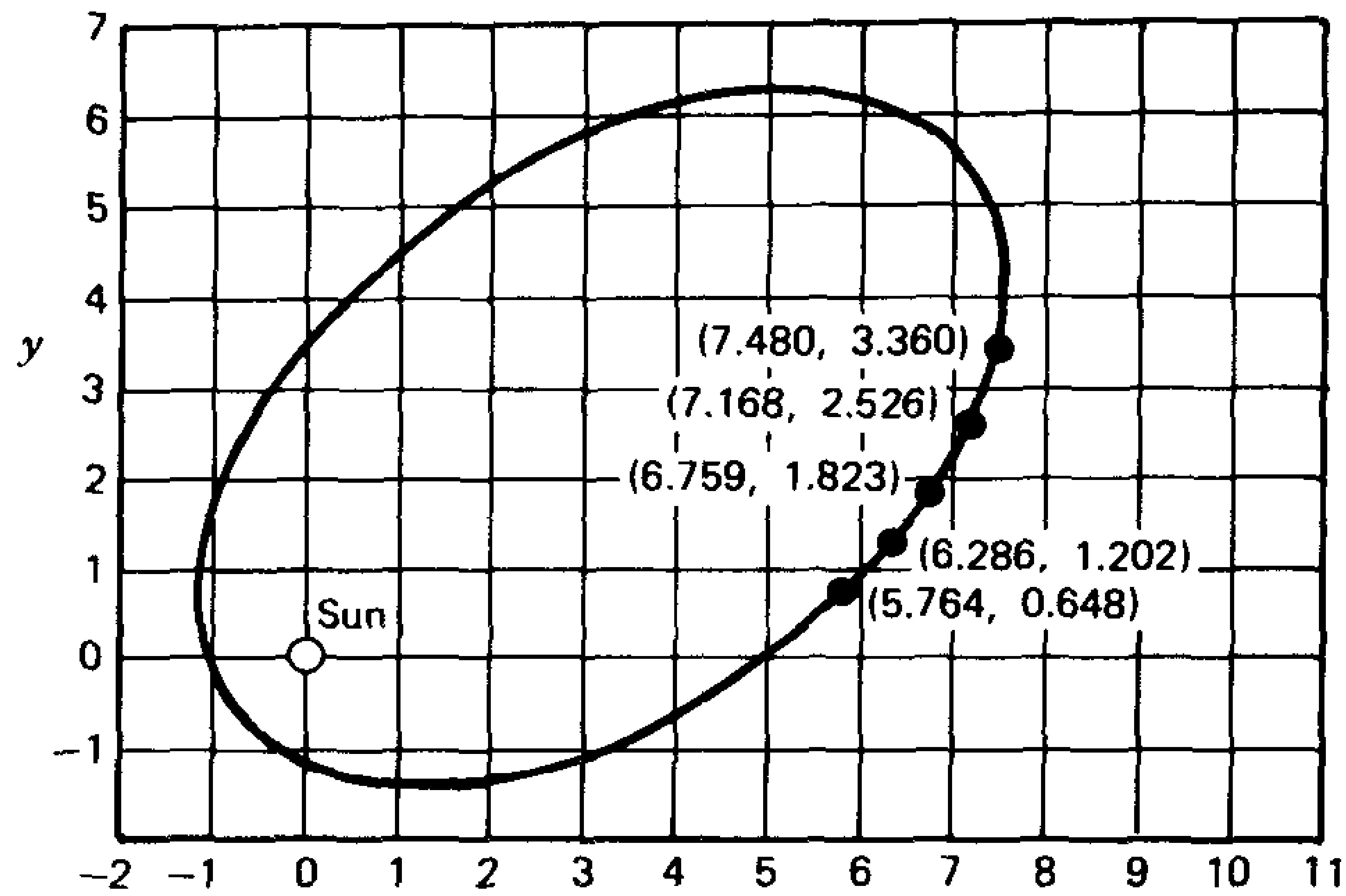


Figure 1.4

The cofactor expansion of this determinant along the first row reduces to

$$x^2 - 1.04xy + 1.30y^2 - 3.90x - 2.93y - 5.49 = 0.$$

Figure 1.4 is a diagram of the orbit, together with the five given points.

A Plane Through Three Points

In Exercise 1.7 we ask the reader to show the following: The plane in 3-space with equation

$$c_1x + c_2y + c_3z + c_4 = 0$$

which passes through three noncollinear points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given by the determinant equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \tag{1.11}$$

Example 1.4 The equation of the plane which passes through the three noncollinear points (1, 1, 0), (2, 0, -1), and (2, 9, 2) is

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 9 & 2 & 1 \end{vmatrix} = 0,$$

which reduces to

$$2x - y + 3z - 1 = 0.$$

A Sphere Through Four Points

In Exercise 1.8 we ask the reader to show the following: The sphere in 3-space with equation

$$c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0$$

which passes through four noncoplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is given by the following determinant equation:

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0. \qquad (1.12)$$

Example 1.5 The equation of the sphere which passes through the four points (0, 3, 2), (1, -1, 1), (2, 1, 0), and (-1, 1, 3) is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ 13 & 0 & 3 & 2 & 1 \\ 3 & 1 & -1 & 1 & 1 \\ 5 & 2 & 1 & 0 & 1 \\ 11 & -1 & 1 & 3 & 1 \end{vmatrix} = 0.$$

This reduces to

$$x^2 + y^2 + z^2 - 4x - 2y - 6z + 5 = 0,$$

which in standard form is

$$(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 9.$$

Exercises

1.1 Find the equations of the lines which pass through the following points:
(a) (1,-1), (2,2)
(b) (0,1), (1,-1).

1.2 Find the equations of the circles which pass through the following points:
(a) (2, 6), (2,0), (5,3)
(b) (2,-2), (3,5), (-4,6).

1.3 Find the equation of the conic section which passes through the points (0,0), (0,-1), (2,0), (2,-5), and (4,-1).

1.4 Find the equations of the planes in 3-space which pass through the following points:
(a) (1,1,-3), (1,-1,1), (0,-1,2)
(b) (2,3,1), (2,-1,-1), (1,2,1).

1.5 Find the equations of the spheres in 3-space which pass through the following points:
(a) (1,2,3), (-1,2,1), (1,0,1), (1,2,-1)
(b) (0,1,-2) (1,3,1) (2,-1,0) (3,1,-1).

1.6 Show that Eq. (1.10) is the equation of the conic section which passes through five given distinct points.

1.7 Show that Eq. (1.11) is the equation of the plane in 3-space which passes through three given noncollinear points.

1.8 Show that Eq. (1.12) is the equation of the sphere in 3-space which passes through four given noncoplanar points.

1.9 Find a determinant equation for the parabola of the form

$$c_1y + c_2x^2 + c_3x + c_4 = 0$$

which passes through three given noncollinear points in the plane.

2

Plane Geometry

Vector algebra is used to prove theorems in plane geometry.

PREREQUISITES:	Plane geometry Vector algebra Dot product

INTRODUCTION

The basic properties of Euclidean space were first systematically studied 23 centuries ago by Euclid, who began with accepted axioms and deduced theorems in plane and solid geometry from them. In this chapter, we prove some elementary theorems in plane geometry using vector algebra. This method usually leads to simpler and more elegant proofs than the methods studied in a first course in geometry — the same methods that Euclid himself employed. This is because the basic equations of vector algebra, such as

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$$

$$\vdots$$

are vector space axioms and therefore apply to Euclidean space. Thus, any new equation we can generate from these basic equations constitutes a *theorem*. Since it is easier to manipulate equations than to write out in words how one is applying axioms, a certain economy of thought is attained and a cleaner presentation of a proof results.

Before presenting some examples of such vector techniques applied to plane geometry, we introduce some notation and review some basic definitions.

Notation and Definitions

For our applications to plane geometry, a vector may be viewed as a directed line segment. We shall use the following notation (Fig. 2.1): If A and B are two points, then AB denotes the (undirected) line segment connecting A and B and $\vec{AB}$ denotes the directed line segment or vector from point A to point B. We shall use $\mathbf{a}$ and $\mathbf{b}$ to denote vectors from an arbitrary fixed reference point O to the points A and B, respectively. Notice that we may write

$$\vec{AB} = \mathbf{b} - \mathbf{a}.$$

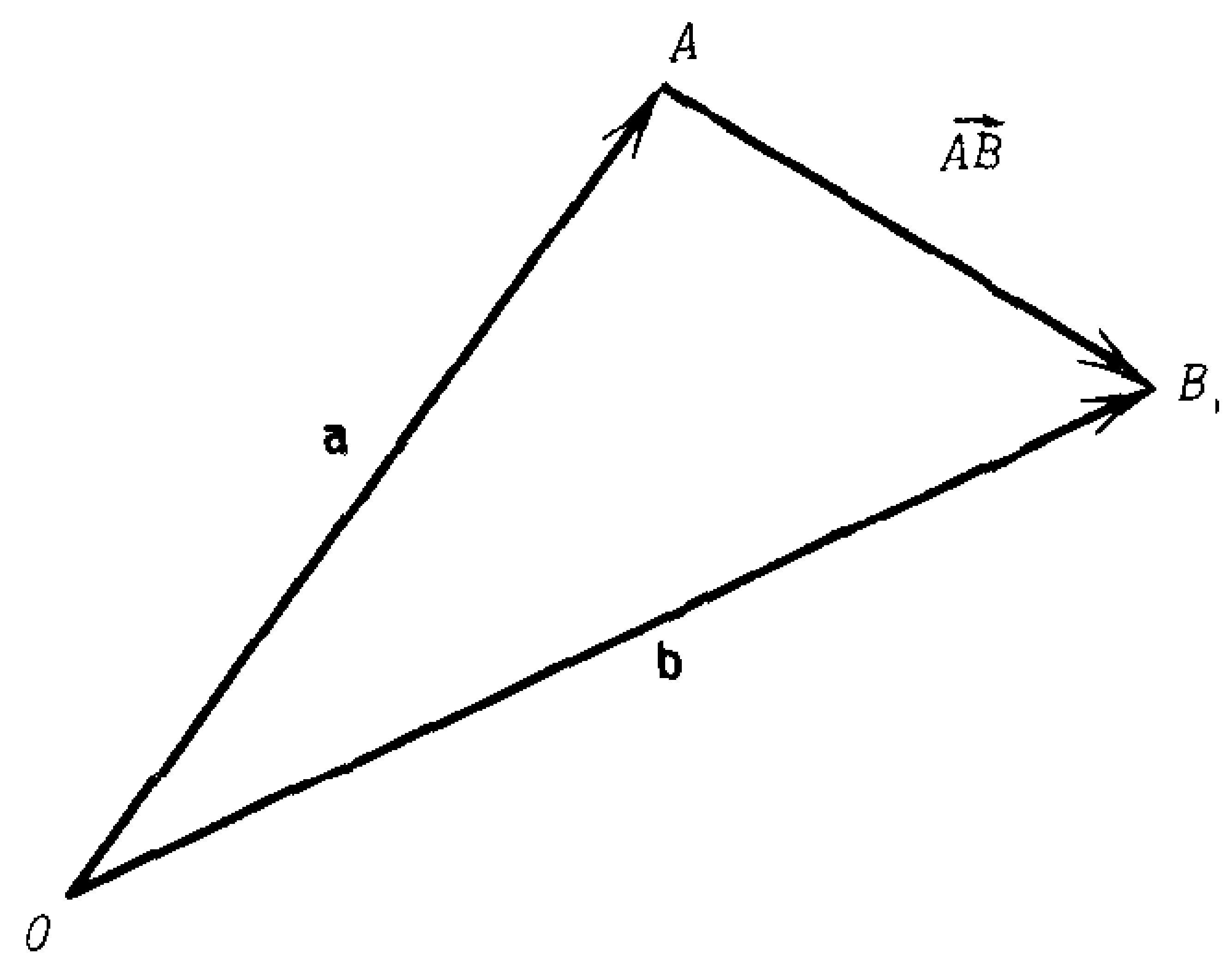


Figure 2.1

We assume that the reader is familiar with the usual rules of vector algebra. However, it will be convenient to state the following criterion for determining when three points are collinear:

Let A, B, and C be three points and let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the vectors from a fixed point O to A, B, and C, respectively (Fig. 2.2). Then B lies on the line segment AC if and only if

$$\mathbf{b} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{c}$$

for some number λ satisfying

$$0 \leq \lambda \leq 1.$$

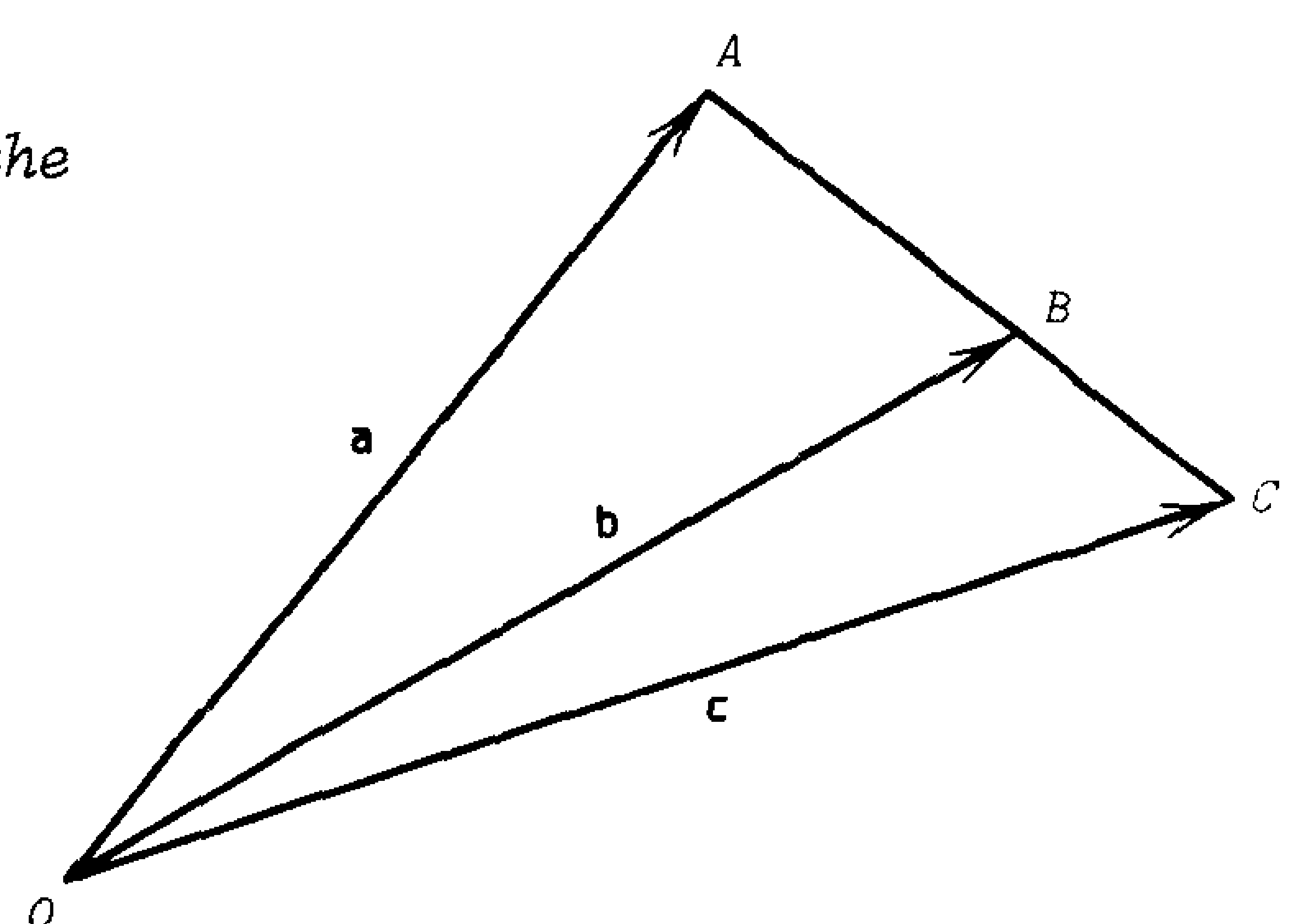


Figure 2.2

When the above criterion is satisfied, the point B cuts the line segment AC in the ratio $\lambda:(1-\lambda)$. For example, if $\lambda = \frac{1}{2}$, i.e. if $\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$, then B is the midpoint of AC; while if $\lambda = \frac{2}{3}$, i.e. if $\mathbf{b} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}$, then B cuts AC in the ratio 2:1.

We also recall the definition of the dot (or scalar or inner) product $\mathbf{a}\cdot\mathbf{b}$ of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta \\ &= ab\cos\theta\end{aligned}$$

Here, $\|\mathbf{a}\|$ or a is the length of vector $\mathbf{a}$, $\|\mathbf{b}\|$ or b is the length of vector $\mathbf{b}$, and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (Fig. 2.3). A useful property of the dot product is the fact that two nonzero vectors are perpendicular to each other if and only if their dot product is zero.

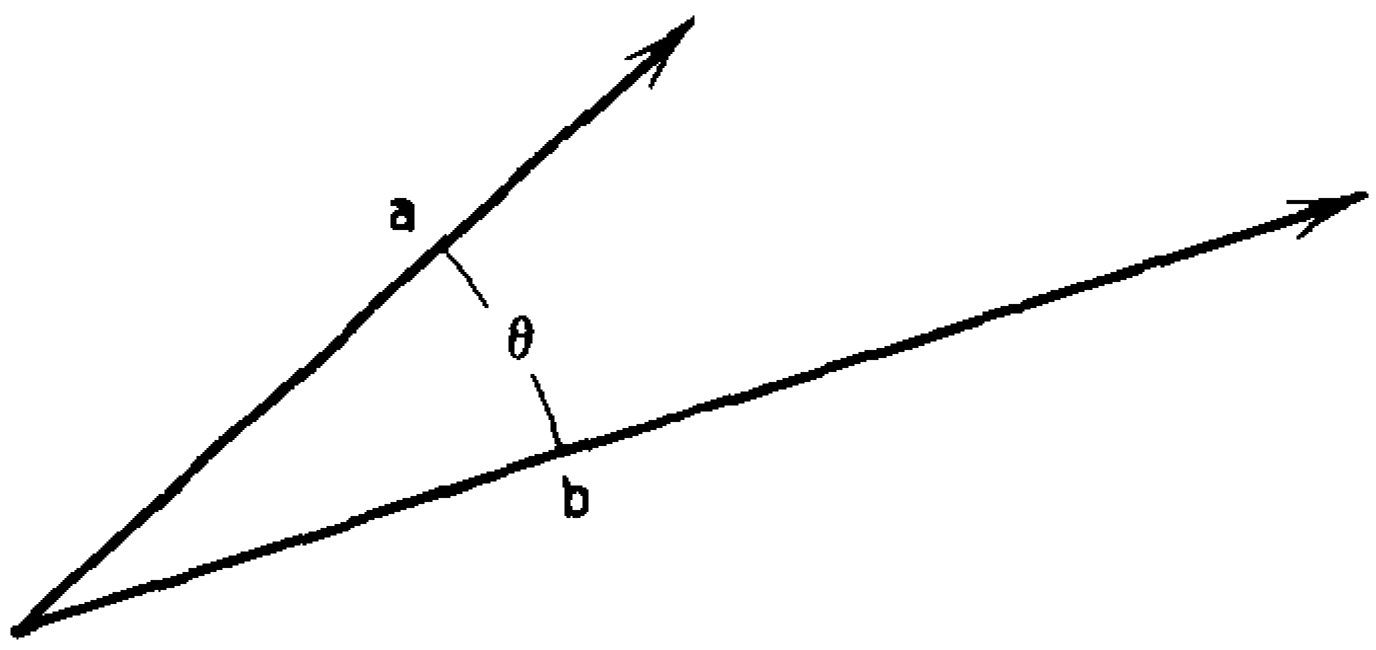
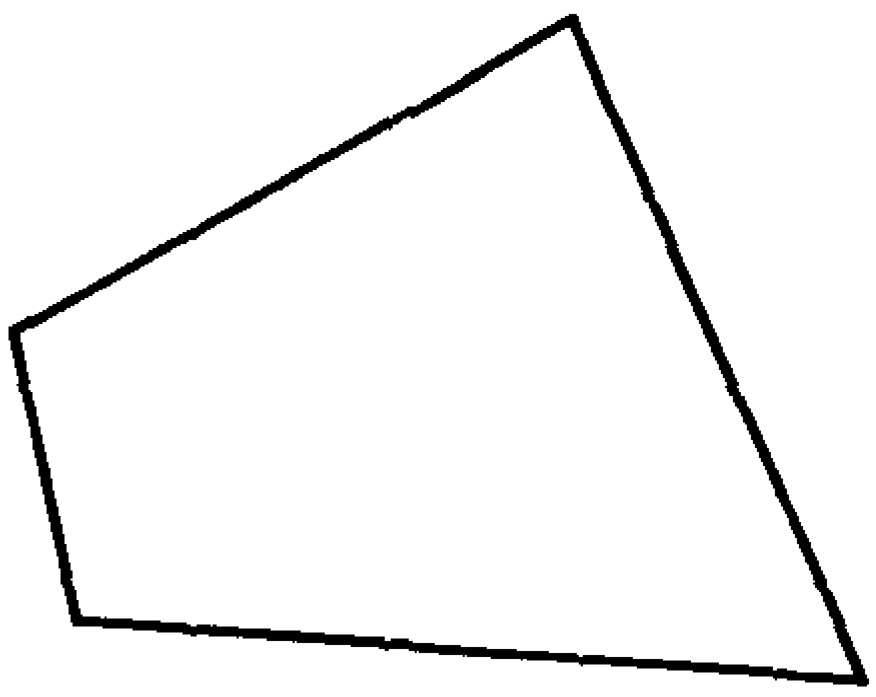
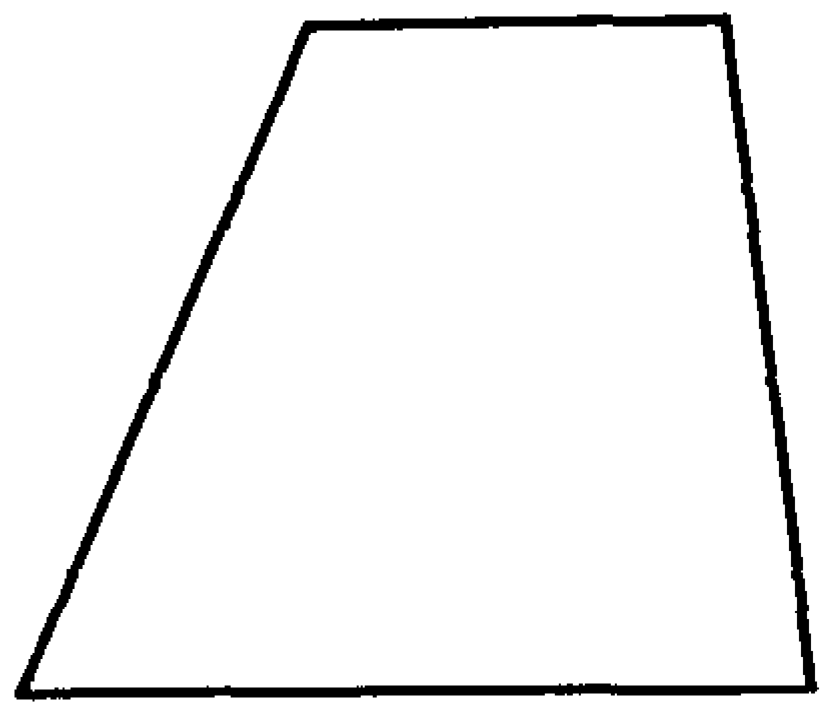


Figure 2.3

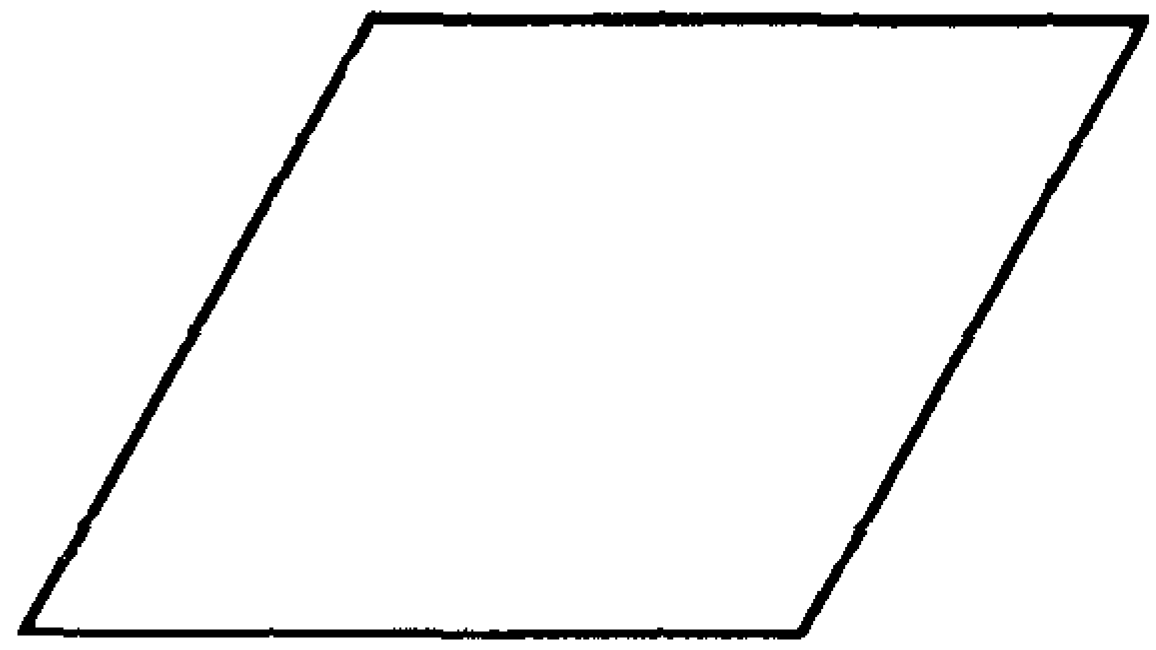
In Fig. 2.4 on the next page, seven definitions from plane geometry are listed for reference. These are followed by six examples in which six basic theorems are proved using vector algebra.



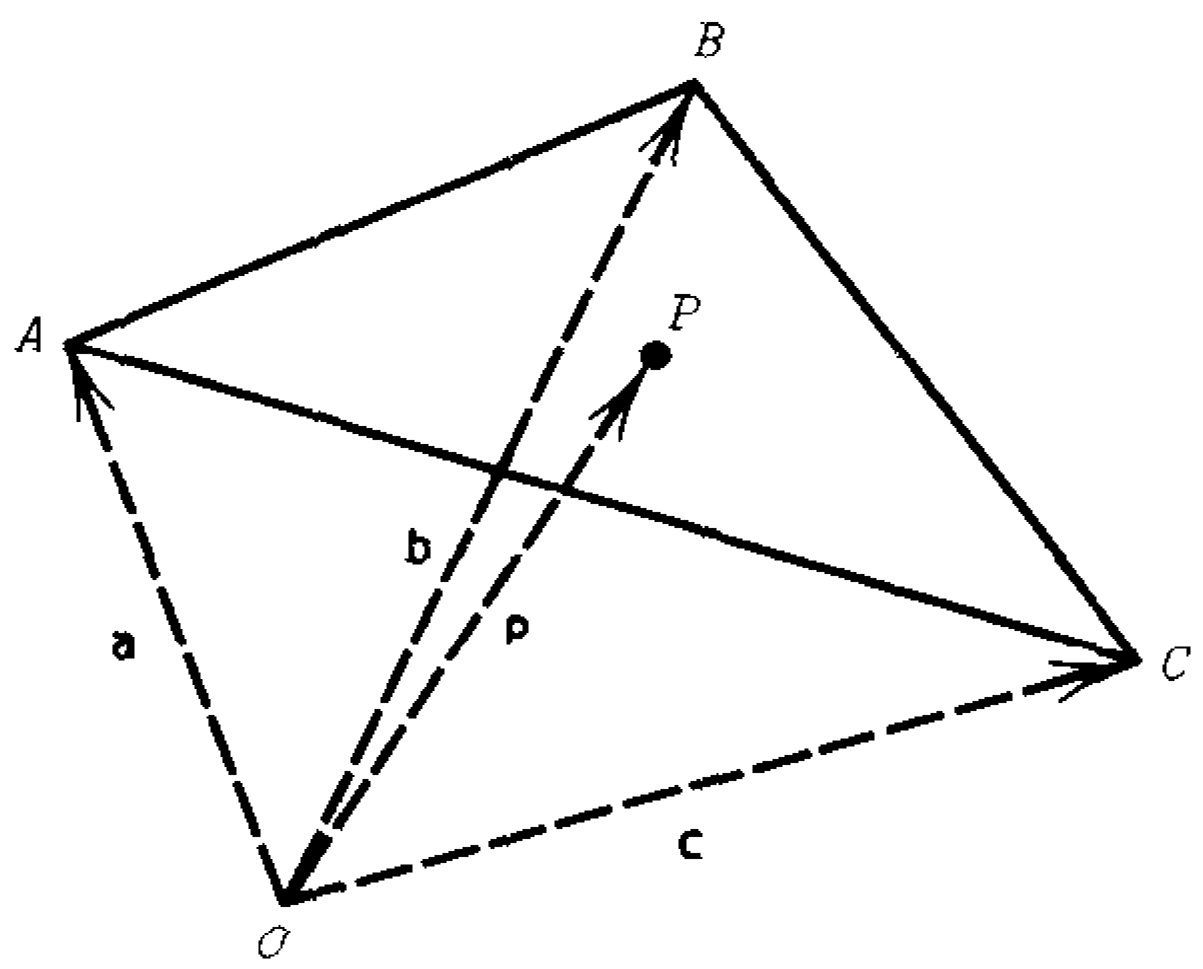
A **quadrilateral** is a four-sided polygon.



A **trapezoid** is a quadrilateral with one pair of parallel sides.

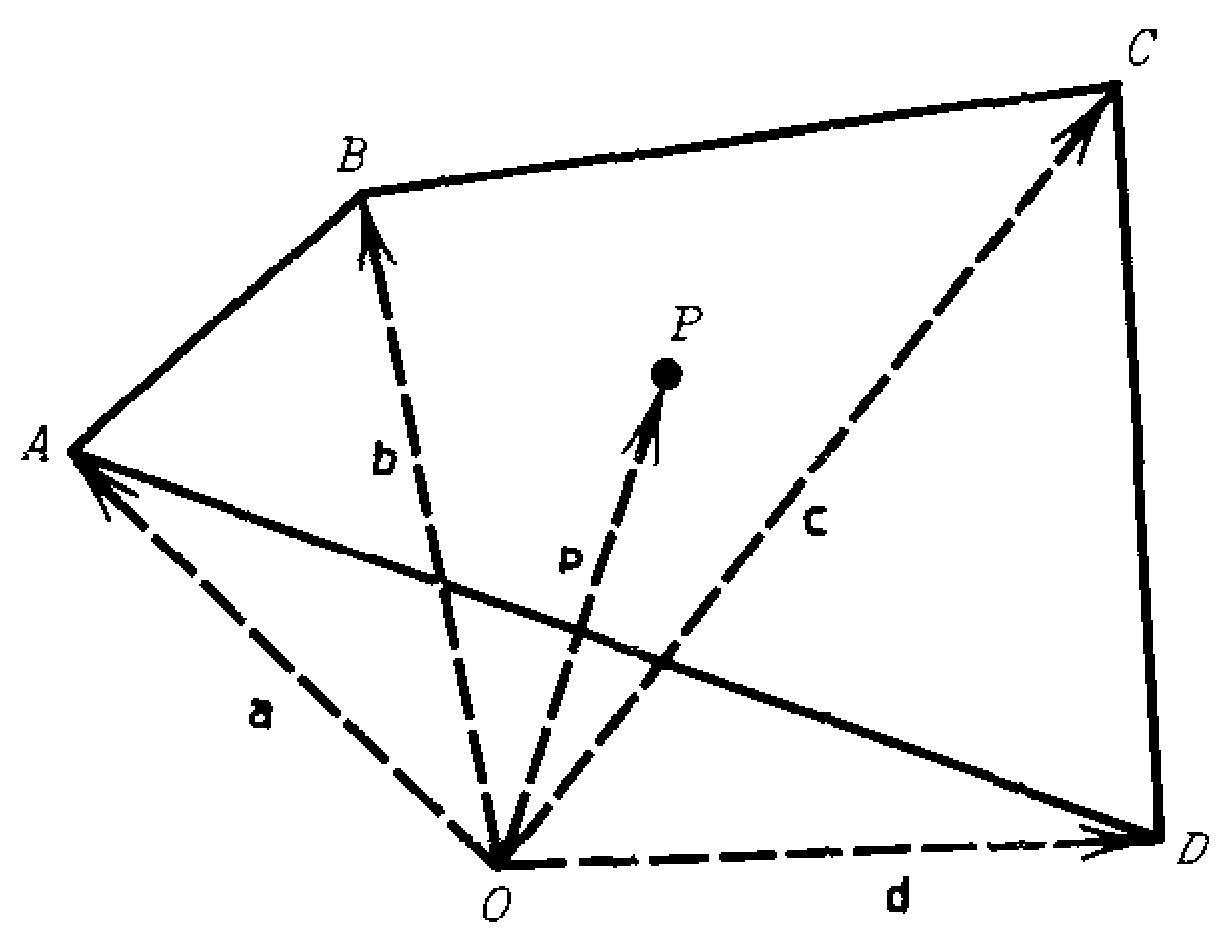


A **parallelogram** is a quadrilateral with two pairs of parallel sides.



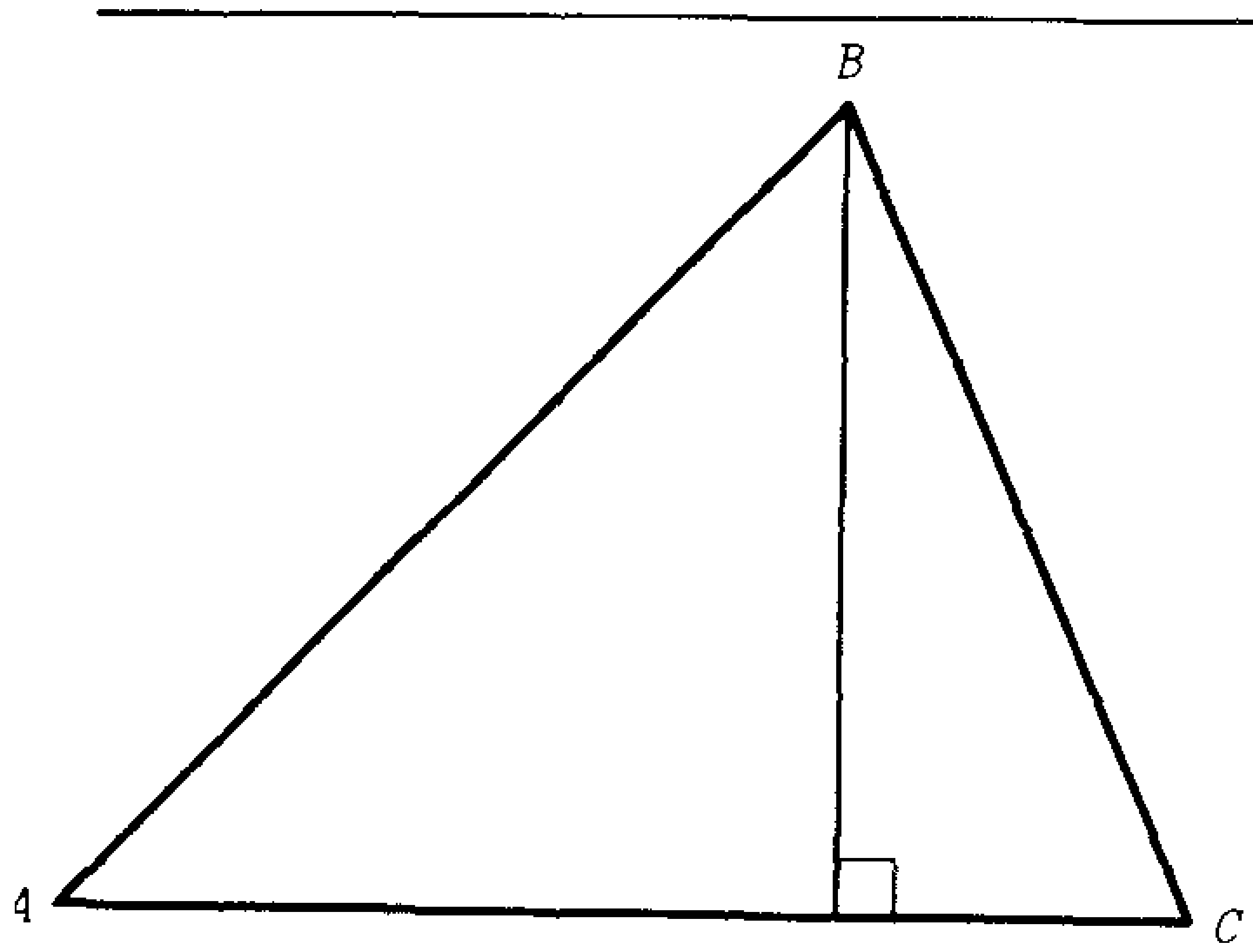
The **centroid or mean center of a triangle** ABC is the point P determined by

$$\mathbf{p} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

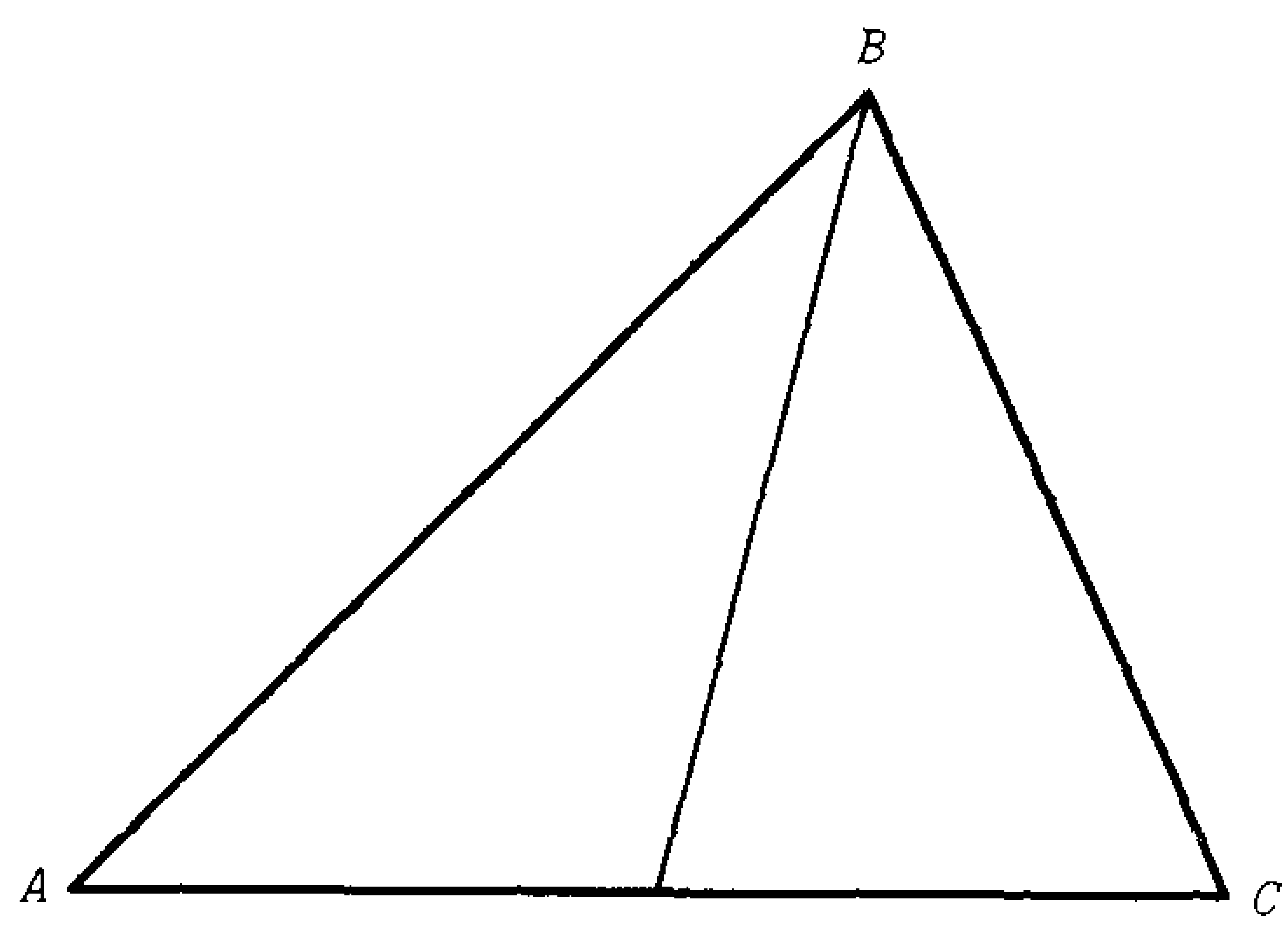


The **centroid or mean center of a quadrilateral** $ABCD$ is the point P determined by

$$\mathbf{p} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$



An **altitude** of a triangle ABC is a line segment from one of the vertices perpendicular to the opposite side.



A **median** of a triangle ABC is a line segment from one of the vertices to the midpoint of the opposide side.

Figure 2.4