Mathematics and Its Applications

R.P. Kuzmina

Asymptotic Methods for Ordinary Differential Equations

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by

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PREFACE

In this book we consider a Cauchy problem for a system of ordinary differential equations with a small parameter. The book is divided into three parts according to three ways of involving the small parameter in the system.

In Part 1 we study *the quasiregular Cauchy problem.* That is, a problem with the singularity included in a bounded function f, which depends on time and a small parameter. This problem is a generalization of the regularly perturbed Cauchy problem studied by Poincaré [35]. Some differential equations which are solved by the averaging method can be reduced to a quasiregular Cauchy problem. As an example, in Chapter 2 we consider the van der Pol problem.

In Part 2 we study *the Tikhonov problem.* This is, a Cauchy problem for a system of ordinary differential equations where the coefficients by the derivatives are integer degrees of a small parameter.

In Part 3 we consider *the double-singular Cauchy problem.* That is, a problem for a system of two vector ordinary differential equations, one of which has a derivative multiplied by an integer degree of a small parameter, and the right hand sides of the equations contain the small parameter in singular, via function f, way (as in Part 1). Thus, the Cauchy problem with a double singularity involves the singularities of both kinds studied in the first two parts of the book. If the differential equation does not depend explicitly on f then the problem becomes a Tikhonov problem from Part 2. In a special case the problem with a double singularity admits equations splitting off some equations which constitute a quasiregular Cauchy problem from Part 1.

For all types of the problems considered in the book we construct series generalizing the well known expansions of Poincaré and Vasiljeva–Imanaliev. We prove that these series are either asymptotic expansions of the solution or else converge to the solution on the interval, on the whole semi-axis, or on asymptotically large time intervals. We prove theorems providing estimates for the remainder term of the asymptotic expansion, the time interval of solution existence and the range of small parameter values. To illustrate the possibilities of the methods considered we give some examples.

The book will be of interest to mathematicians specialising in differential equations and to applied mathematicians who use the asymptotic methods for ordinary differential equations.

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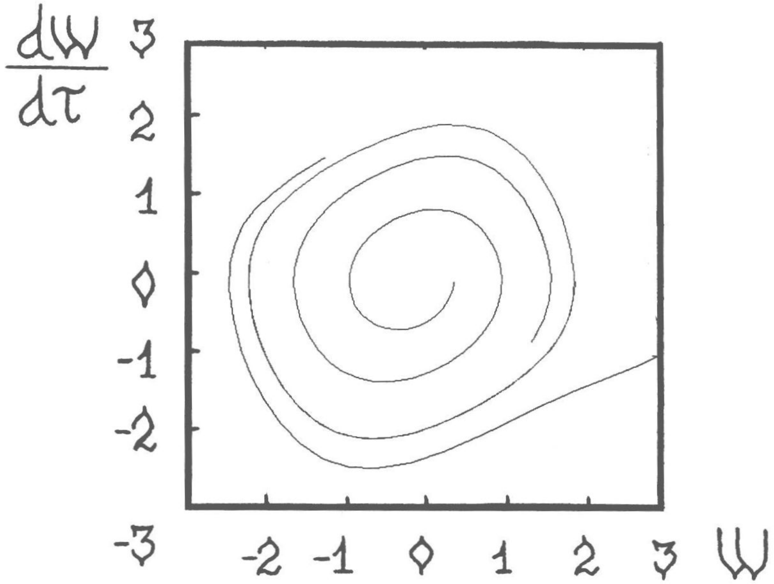

The van der Pol problem.
The phase plane. $\varepsilon = 0.2$.

CHAPTER 1

SOLUTION EXPANSIONS OF THE QUASIREGULAR CAUCHY PROBLEM

§1. Solution of the Quasiregular Cauchy Problem

1.1. DEFINITION OF THE QUASIREGULAR CAUCHY PROBLEM

Consider a Cauchy problem

$$\frac{dx}{dt} = F(x, t, \varepsilon, f(t, \varepsilon)), \qquad x|_{t=0} = 0. \tag{1.1}$$

Here $x \in \mathbf{R}^N$, $F \in \mathbf{R}^N$ are N-dimensional vectors, $f \in \mathbf{R}^M$ is an M-dimensional vector, $\varepsilon \in \mathbf{R}$ is a small parameter, $t \in \mathbf{R}$ is an independent variable (time), $\mathbf{R}^N$ is N-dimensional real space.

Assume some notation. Let $D_x \subset \mathbf{R}^N$ be a neighborhood of the point $x = 0$, $D_f \subset \mathbf{R}^{\mathbf{M}}$ be a bounded domain, T, $\overline{\varepsilon}$ be positive numbers.

Definition 1.1. *The problem* (1.1) *is called a quasiregular Cauchy problem if: 1)* $F(x, t, \varepsilon, f)$ *is a smooth function defined on the direct product of the neighborhood* D_x*, the intervals* $0 \leq t \leq T$, $0 \leq \varepsilon \leq \overline{\varepsilon}$ *and the domain* D_f*; 2)* f *is a smooth function mapping the direct product of the intervals* $0 \leq t \leq T$, $0 < \varepsilon \leq \overline{\varepsilon}$ *to the domain* D_f.

If the right side of the differential equation (1.1) does not depend explicitly on f, then (1.1) is a regularly perturbed Cauchy problem studied by Poincaré [35] (see §9). As an example of function f complying with Definition 1.1, we can take

$$f = \Big(\exp(-t/\varepsilon),\ \cos(t/\varepsilon)\Big).$$

1.2. CONSTRUCTION OF THE SOLUTION

Consider a problem with two small parameters:

$$\frac{dz}{dt} = F(z, t, \varepsilon, f(t, \mu)), \qquad z|_{t=0} = 0. \tag{1.2}$$

The problem (1.2) is a regularly perturbed Cauchy problem with respect to the parameter ε. Its solution is constructed by *Poincaré's method of the small parameter*, which goes as follows (suppose all the operations make sense) [35].

- Let the solution $z = z(t, \varepsilon, \mu)$ be expanded in powers of the small parameter ε:

$$z(t, \varepsilon, \mu) = \sum_{k=0}^{\infty} z^{(k)}(t, \mu)\, \varepsilon^k. \tag{1.3}$$

- Substitute (1.3) in the equation (1.2).
- Expand both sides of the equations (1.2) in powers of ε.
- Set equal the coefficients in terms with equal powers of ε.

Finally, we have the equations for coefficients of the series (1.3).

The coefficient $z^{(0)}(t, \mu)$ (*zero approximation of the solution* $z(t, \varepsilon, \mu)$ of Eq. (1.2)) is the solution of the *degenerate problem*

$$\frac{dz^{(0)}}{dt} = F(z^{(0)}, t, 0, f(t, \mu)), \qquad z^{(0)}|_{t=0} = 0. \tag{1.4}$$

For any $k \geq 0$ the coefficient $z^{(k)}(t, \mu)$ is a solution of the equation

$$\frac{dz^{(k)}}{dt} = \left[F\Big(\sum_{i=0}^{k} z^{(i)}(t, \mu)\, \varepsilon^i,\ t,\ \varepsilon, f(t, \mu) \Big) \right]^{(k)}; \qquad z^{(k)}|_{t=0} = 0. \tag{1.5}$$

Here, by square brackets with the upper index $^{(k)}$ we denote the coefficient of the term with ε^k in the power expansion of the function F with respect to ε. Write the equations (1.5) for $k \geq 1$ in the form

$$\frac{dz^{(k)}}{dt} = A(t, \mu)\, z^{(k)} + F^{(k)}(t, \mu), \qquad z^{(k)}|_{t=0} = 0. \tag{1.6}$$

Here we put by definition

$$\begin{aligned} A(t, \mu) &\equiv F_x(z^{(0)}(t, \mu), t, 0, f(t, \mu)), \\ F^{(k)}(t, \mu) &\equiv \left[F\Big(\sum_{i=0}^{k-1} z^{(i)}(t, \mu)\, \varepsilon^i,\ t,\ \varepsilon, f(t, \mu) \Big) \right]^{(k)}, \end{aligned} \tag{1.7}$$

F_x is *a Jacobi matrix* of partial derivatives of the vector F with respect to components of the vector x. The function $F^{(k)}(t, \mu)$ depends on $z^{(0)}(t, \mu)$, $\dots$, $z^{(k-1)}(t, \mu)$, $k \geq 1$.

The problem (1.6) is linear. Its solution has the form

$$z^{(k)}(t,\mu) = \int_0^t U(t,s,\mu) \cdot F^{(k)}(s,\mu)\, ds. \tag{1.8}$$

Here $U(t,s,\mu)$ is *the Cauchy matrix of the variational equation*

$$\frac{d\zeta}{dt} = A(t,\mu)\,\zeta. \tag{1.9}$$

(A Cauchy matrix $U(t,s,\mu)$ is the fundamental matrix of the system (1.9). It equals the identity matrix when $t=s$: $U(s,s,\mu)=E$).

When $\mu=\varepsilon$, the expansion (1.3) takes the form

$$\boxed{x(t,\varepsilon) = \sum_{k=0}^{\infty} z^{(k)}(t,\varepsilon)\,\varepsilon^k.} \tag{1.10}$$

In Theorems 2.1–2.4 below we prove that the series (1.10) converges to the solution of the problem (1.1). Under conditions of Theorems 2.5–2.8, the series (1.10) is an asymptotic expansion for the solution of the problem (1.1):

$$\boxed{x(t,\varepsilon) \sim \sum_{k=0}^{\infty} z^{(k)}(t,\varepsilon)\,\varepsilon^k} \tag{1.11}$$

(see in Subsection 2.2 the definition and notation for an asymptotic expansion).

Thus, in order to find the solution in the form (1.10), (1.11) we must know the zero approximation $z^{(0)}(t,\mu)$ of the solution and the matrix $U(t,s,\mu)$. Then we can successively compute the coefficients of the expansions (1.10), (1.11) for $k=1, 2, \ldots$ using the formulas (1.8).

In some cases the following theorem helps to find the Cauchy matrix explicitly.

Theorem 1.1. (Poincaré [4]**)** *If the general solution $g(t,C)$ of the differential equations*

$$\frac{dx}{dt} = F(x,t) \tag{1.12}$$

is known (C is a vector of arbitrary constants), then the Cauchy matrix of the system

$$\frac{d\zeta}{dt} = F_x(x^{(0)}(t),t)\,\zeta$$

has the form

$$U(t,s) = U_*(t)\cdot U_*^{-1}(s), \qquad U_*(t) = \left(\frac{\partial g}{\partial C_1} \;\cdots\; \frac{\partial g}{\partial C_N}\right)\Big|_{C=C^\circ}.$$

Here $x^{(0)}(t)$ is a partial solution of the problem (1.12) and C° is a constant corresponding to this partial solution: $g(t, C^\circ) = x^{(0)}(t)$.

In (1.12) the functions and the solution may depend on parameters. Thus, the Cauchy matrix $U(t, s, \mu)$ for the equation (1.9) can be found if we know the general solution of the equation (1.4).

1.3. REDUCTION OF THE NONZERO INITIAL VALUES PROBLEM TO (1.1)

Now we are going to show how to reduce the problem

$$\frac{d\widetilde{x}}{dt} = \widetilde{F}(\widetilde{x}, t, \varepsilon, \widetilde{f}(t, \varepsilon)), \qquad \widetilde{x}|_{t=0} = \widetilde{x}^\circ(\varepsilon) \tag{1.13}$$

to the problem (1.1). (Suppose all the operations make sense). If the initial values are smooth, then the degenerate problem for (1.13) has the form

$$\frac{d\widetilde{z}^{(0)}}{dt} = \widetilde{F}(\widetilde{z}^{(0)}, t, 0, \widetilde{f}(t, \mu)), \qquad \widetilde{z}^{(0)}|_{t=0} = \widetilde{x}^\circ(0). \tag{1.14}$$

Let us introduce new variables

$$x = \widetilde{x} - \widetilde{z}^{(0)}(t, \varepsilon) - \widetilde{x}^\circ(\varepsilon) + \widetilde{x}^\circ(0). \tag{1.15}$$

From Eqs. (1.13), (1.14) it follows that x is the solution of the problem (1.1) with

$$\begin{aligned} F(x, t, \varepsilon, f(t, \varepsilon)) \;&\equiv\; \widetilde{F}\Big(x + \widetilde{z}^{(0)}(t, \varepsilon) + \widetilde{x}^\circ(\varepsilon) - \widetilde{x}^\circ(0),\ t,\ \varepsilon,\ \widetilde{f}(t, \varepsilon)\Big) \\ &\quad - \widetilde{F}\Big(\widetilde{z}^{(0)}(t, \varepsilon), t, 0, \widetilde{f}(t, \varepsilon)\Big) \end{aligned} \tag{1.16}$$

This completes the proof. Note that if the function f can be expressed via $\widetilde{z}^{(0)}$, $\widetilde{f}$, that is, if

$$\begin{aligned} F(x, t, \varepsilon, f(t, \mu)) \;&=\; \widetilde{F}\Big(x + \widetilde{z}^{(0)}(t, \mu) + \widetilde{x}^\circ(\varepsilon) - \widetilde{x}^\circ(0), t, \varepsilon, \widetilde{f}(t, \mu)\Big) \\ &\quad - \widetilde{F}\Big(\widetilde{z}^{(0)}(t, \mu), t, 0, \widetilde{f}(t, \mu)\Big), \end{aligned}$$

then (1.13) takes the form of the problem (1.1) with the condition

$$F(0, t, 0, f(t, \mu)) = 0$$

(see Condition 2.1). In this case the zero approximation of the solution of the problem (1.1) is trivial: $z^{(0)}(t, \varepsilon) = 0$ ($z^{(0)}(t, \mu)$ is the solution of the problem (1.4)).