

国外数学名著系列 (续一)

(影印版) 50

V. I. Arnol'd (Ed.)

Dynamical Systems V

Bifurcation Theory and Catastrophe Theory

动力系统 V

分岐理论和突变理论



科学出版社
www.sciencep.com

国外数学名著系列(影印版) 50

Dynamical Systems V

Bifurcation Theory and Catastrophe Theory

动力系统 V

分岐理论和突变理论

V. I. Arnol'd (Ed.)

科学出版社

北京

图字: 01-2008-5383

V. I. Arnol'd(Ed.): Dynamical Systems V: Bifurcation Theory and Catastrophe Theory

© Springer-Verlag Berlin Heidelberg 1994

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the People's Republic of China only and not for export therefrom

本书英文影印版由德国施普林格出版公司授权出版。未经出版者书面许可,不得以任何方式复制或抄袭本书的任何部分。本书仅限在中华人民共和国销售,不得出口。版权所有,翻印必究。

图书在版编目(CIP)数据

动力系统 V: 分歧理论和突变理论=Dynamical Systems V: Bifurcation Theory and Catastrophe Theory / (俄罗斯) 阿诺德(Arnol'd, V. I.) 编著. —影印版. —北京: 科学出版社, 2009

(国外数学名著系列; 50)

ISBN 978-7-03-023493-3

I. 动… II. 阿… III. ①动力系统(数学)-英文 ②分歧理论-英文
③突变理论-英文 IV. O175

中国版本图书馆 CIP 数据核字(2008) 第 186185 号

责任编辑: 范庆奎 / 责任印刷: 钱玉芬 / 封面设计: 黄华斌

科学出版社出版

北京东黄城根北街 16 号

邮政编码: 100717

<http://www.sciencep.com>

双青印刷厂印刷

科学出版社发行 各地新华书店经销

*

2009 年 1 月第 一 版 开本: B5(720×1000)

2009 年 1 月第一次印刷 印张: 18

印数: 1—2 500 字数: 341 000

定价: 68.00 元

(如有印装质量问题, 我社负责调换〈双青〉)

《国外数学名著系列》(影印版)专家委员会

(按姓氏笔画排序)

丁伟岳 王 元 文 兰 石钟慈 冯克勤 严加安
李邦河 李大潜 张伟平 张继平 杨 乐 姜伯驹
郭 雷

项目策划

向安全 林 鹏 王春香 吕 虹 范庆奎 王 璐

执行编辑

范庆奎

《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来,需要数学家淡泊名利并付出更艰苦地努力。另一方面,我们也要从客观上为数学家创造更有利的发展数学事业的外部环境,这主要是加强对数学事业的支持与投资力度,使数学家有较好的工作与生活条件,其中也包括改善与加强数学的出版工作。

从出版方面来讲,除了较好较快地出版我们自己的成果外,引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说,施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书,使我国广大数学家能以较低的价格购买,特别是在边远地区工作的数学家能普遍见到这些书,无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权,一次影印了 23 本施普林格出版社出版的数学书,就是一件好事,也是值得继续做下去的事情。大体上分一下,这 23 本书中,包括基础数学书 5 本,应用数学书 6 本与计算数学书 12 本,其中有些书也具有交叉性质。这些书都是很新的,2000 年以后出版的占绝大部分,共计 16 本,其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿,例如基础数学中的数论、代数与拓扑三本,都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点,基础数学类的书以“经典”为主,应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家,例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士,曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然,23 本书只能涵盖数学的一部分,所以,这项工作还应该继续做下去。更进一步,有些读者面较广的好书还应该翻译成中文出版,使之有更大的读者群。

总之,我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持,并盼望这一工作取得更大的成绩。

王 元

2005 年 12 月 3 日

List of Editors, Authors and Translators

Editor-in-Chief

R.V. Gamkrelidze, Russian Academy of Sciences, Steklov Mathematical Institute, ul. Vavilova 42, 117966 Moscow, Institute for Scientific Information (VINITI), ul. Usievicha 20a, 125219 Moscow, Russia

Consulting Editor

V.I. Arnol'd, Steklov Mathematical Institute, ul. Vavilova 42, 117966 Moscow, Russia

Authors

V.S. Afrajmovich, N. Novgorod State University, pl. Gagarina 23, 603078 N. Novgorod, Russia

V.I. Arnol'd, Steklov Mathematical Institute, ul. Vavilova 42, 117966 Moscow, Russia

Yu.S. Il'yashenko, Department of Mathematics and Mechanics, Moscow State University, 119899 Moscow, Russia

L.P. Shil'nikov, Institute for Applied Mathematics and Cybernetics, ul. Ul'janova 10, 603005 Nizhnij Novgorod, Russia

Translator

N.D. Kazarinoff†

Translator's Preface

In translating this volume, I am happy to thank Y.-H. Wan and James Boa for much help on technical points and P. Ashwin for the final check of Part I. I am particularly thankful to G. Wassermann for his careful reading of and many excellent suggestions for the translation of Part II.

N.D. Kazarinoff

Acknowledgement

Springer-Verlag would like to thank J. Joel, B. Khesin, V. Arnol'd and A. Paice for their mathematical and linguistic editing which was necessary after the untimely death of N.D. Kazarinoff. Without their efforts this book would have been delayed even longer.

Springer-Verlag, September 1993

Contents

I. Bifurcation Theory

V.I. Arnol'd, V.S. Afrajmovich,
Yu.S. Il'yashenko, L.P. Shil'nikov

1

II. Catastrophe Theory

V.I. Arnol'd

207

Author Index

265

Subject Index

269

I. Bifurcation Theory

V.I. Arnol'd, V.S. Afraimovich,
Yu. S. Il'yashenko, L.P. Shil'nikov

Translated from the Russian
by N.D. Kazarinoff

Contents

| | |
|--|----|
| Preface | 7 |
| Chapter 1. Bifurcations of Equilibria | 10 |
| § 1. Families and Deformations | 11 |
| 1.1. Families of Vector Fields | 11 |
| 1.2. The Space of Jets | 11 |
| 1.3. Sard's Lemma and Transversality Theorems | 12 |
| 1.4. Simplest Applications: Singular Points of Generic Vector Fields | 13 |
| 1.5. Topologically Versal Deformations | 14 |
| 1.6. The Reduction Theorem | 15 |
| 1.7. Generic and Principal Families | 16 |
| § 2. Bifurcations of Singular Points in Generic One-Parameter Families | 17 |
| 2.1. Typical Germs and Principal Families | 17 |
| 2.2. Soft and Hard Loss of Stability | 19 |
| § 3. Bifurcations of Singular Points in Generic Multi-Parameter Families with Simply Degenerate Linear Parts | 20 |
| 3.1. Principal Families | 20 |
| 3.2. Bifurcation Diagrams of the Principal Families (3^\pm) in Table 1 | 21 |
| 3.3. Bifurcation Diagrams with Respect to Weak Equivalence and Phase Portraits of the Principal Families (4^\pm) in Table 1 | 21 |
| § 4. Bifurcations of Singular Points of Vector Fields with a Doubly-Degenerate Linear Part | 23 |
| 4.1. A List of Degeneracies | 23 |
| 4.2. Two Zero Eigenvalues | 24 |
| 4.3. Reductions to Two-Dimensional Systems | 24 |
| 4.4. One Zero and a Pair of Purely Imaginary Eigenvalues | 25 |
| 4.5. Two Purely Imaginary Pairs | 29 |

| | |
|---|----|
| 4.6. Principal Deformations of Equations of Difficult Type in Problems with Two Pairs of Purely Imaginary Eigenvalues (Following Żoladek) | 33 |
| § 5. The Exponents of Soft and Hard Loss of Stability | 35 |
| 5.1. Definitions | 35 |
| 5.2. Table of Exponents | 37 |
| Chapter 2. Bifurcations of Limit Cycles | 38 |
| § 1. Bifurcations of Limit Cycles in Generic One-Parameter Families .. | 39 |
| 1.1. Multiplier 1 | 39 |
| 1.2. Multiplier -1 and Period-Doubling Bifurcations | 41 |
| 1.3. A Pair of Complex Conjugate Multipliers | 42 |
| 1.4. Nonlocal Bifurcations in One-Parameter Families of Diffeomorphisms | 43 |
| 1.5. Nonlocal Bifurcations of Periodic Solutions | 45 |
| 1.6. Bifurcations Resulting in Destructions of Invariant Tori | 45 |
| § 2. Bifurcations of Cycles in Generic Two-Parameter Families with an Additional Simple Degeneracy | 48 |
| 2.1. A List of Degeneracies | 48 |
| 2.2. A Multiplier $+1$ or -1 with Additional Degeneracy in the Nonlinear Terms | 49 |
| 2.3. A Pair of Multipliers on the Unit Circle with Additional Degeneracy in the Nonlinear Terms | 49 |
| § 3. Bifurcations of Cycles in Generic Two-Parameter Families with Strong Resonances of Orders $q \neq 4$ | 51 |
| 3.1. The Normal Form in the Case of Unipotent Jordan Blocks | 51 |
| 3.2. Averaging in the Seifert and the Möbius Foliations | 52 |
| 3.3. Principal Vector Fields and their Deformations | 53 |
| 3.4. Versality of Principal Deformations | 53 |
| 3.5. Bifurcations of Stationary Solutions of Periodic Differential Equations with Strong Resonances of Orders $q \neq 4$ | 54 |
| § 4. Bifurcations of Limit Cycles for a Pair of Multipliers Crossing the Unit Circle at $\pm i$ | 57 |
| 4.1. Degenerate Families | 57 |
| 4.2. Degenerate Families Found Analytically | 59 |
| 4.3. Degenerate Families Found Numerically | 59 |
| 4.4. Bifurcations in Nondegenerate Families | 60 |
| 4.5. Limit Cycles of Systems with a Fourth Order Symmetry | 60 |
| § 5. Finitely-Smooth Normal Forms of Local Families | 60 |
| 5.1. A Synopsis of Results | 60 |
| 5.2. Definitions and Examples | 62 |
| 5.3. General Theorems and Deformations of Nonresonant Germs .. | 63 |
| 5.4. Reduction to Linear Normal Form | 65 |
| 5.5. Deformations of Germs of Diffeomorphisms of Poincaré Type | 66 |

| | |
|--|-----|
| 5.6. Deformations of Simply Resonant Hyperbolic Germs | 66 |
| 5.7. Deformations of Germs of Vector Fields with One Zero Eigenvalue at a Singular Point | 68 |
| 5.8. Functional Invariants of Diffeomorphisms of the Line | 69 |
| 5.9. Functional Invariants of Local Families of Diffeomorphisms .. | 70 |
| 5.10. Functional Invariants of Families of Vector Fields | 71 |
| 5.11. Functional Invariants of Topological Classifications of Local Families of Diffeomorphisms of the Line | 71 |
| § 6. Feigenbaum Universality for Diffeomorphisms and Flows | 73 |
| 6.1. Period-Doubling Cascades | 73 |
| 6.2. Perestroikas of Fixed Points | 75 |
| 6.3. Cascades of n -fold Increases of Period | 75 |
| 6.4. Doubling in Hamiltonian Systems | 75 |
| 6.5. The Period-Doubling Operator for One-Dimensional Mappings | 75 |
| 6.6. The Universal Period-Doubling Mechanism for Diffeomorphisms | 77 |
| Chapter 3. Nonlocal Bifurcations | 79 |
| § 1. Degeneracies of Codimension 1. Summary of Results | 80 |
| 1.1. Local and Nonlocal Bifurcations | 80 |
| 1.2. Nonhyperbolic Singular Points | 82 |
| 1.3. Nonhyperbolic Cycles | 83 |
| 1.4. Nontransversal Intersections of Manifolds | 84 |
| 1.5. Contours | 85 |
| 1.6. Bifurcation Surfaces | 87 |
| 1.7. Characteristics of Bifurcations | 88 |
| 1.8. Summary of Results | 88 |
| § 2. Nonlocal Bifurcations of Flows on Two-Dimensional Surfaces | 90 |
| 2.1. Semilocal Bifurcations of Flows on Surfaces | 90 |
| 2.2. Nonlocal Bifurcations on a Sphere: The One-Parameter Case .. | 91 |
| 2.3. Generic Families of Vector Fields | 92 |
| 2.4. Conditions for Genericity | 94 |
| 2.5. One-Parameter Families on Surfaces different from the Sphere | 95 |
| 2.6. Global Bifurcations of Systems with a Global Transversal Section on a Torus | 96 |
| 2.7. Some Global Bifurcations on a Klein bottle | 97 |
| 2.8. Bifurcations on a Two-Dimensional Sphere: The Multi-Parameter Case | 98 |
| 2.9. Some Open Questions | 101 |
| § 3. Bifurcations of Trajectories Homoclinic to a Nonhyperbolic Singular Point | 102 |
| 3.1. A Node in its Hyperbolic Variables | 103 |
| 3.2. A Saddle in its Hyperbolic Variables: One Homoclinic Trajectory | 103 |

| | |
|--|-----|
| 3.3. The Topological Bernoulli Automorphism | 104 |
| 3.4. A Saddle in its Hyperbolic Variables: Several Homoclinic Trajectories | 105 |
| 3.5. Principal Families | 106 |
| § 4. Bifurcations of Trajectories Homoclinic to a Nonhyperbolic Cycle | 106 |
| 4.1. The Structure of a Family of Homoclinic Trajectories | 107 |
| 4.2. Critical and Noncritical Cycles | 107 |
| 4.3. Creation of a Smooth Two-Dimensional Attractor | 108 |
| 4.4. Creation of Complex Invariant Sets (The Noncritical Case) ... | 109 |
| 4.5. The Critical Case | 109 |
| 4.6. A Two-Step Transition from Stability to Turbulence | 111 |
| 4.7. A Noncompact Set of Homoclinic Trajectories | 112 |
| 4.8. Intermittency | 113 |
| 4.9. Accessibility and Nonaccessibility | 113 |
| 4.10. Stability of Families of Diffeomorphisms | 114 |
| 4.11. Some Open Questions | 116 |
| § 5. Hyperbolic Singular Points with Homoclinic Trajectories | 116 |
| 5.1. Preliminary Notions: Leading Directions and Saddle Numbers | 117 |
| 5.2. Bifurcations of Homoclinic Trajectories of a Saddle that Take Place on the Boundary of the Set of Morse-Smale Systems ... | 117 |
| 5.3. Requirements for Genericity | 118 |
| 5.4. Principal Families in \mathbb{R}^3 and their Properties | 119 |
| 5.5. Versality of the Principal Families | 122 |
| 5.6. A Saddle with Complex Leading Direction in \mathbb{R}^3 | 122 |
| 5.7. An Addition: Bifurcations of Homoclinic Loops Outside the Boundary of a Set of Morse-Smale Systems | 126 |
| 5.8. An Addition: Creation of a Strange Attractor upon Bifurcation of a Trajectory Homoclinic to a Saddle | 127 |
| § 6. Bifurcations Related to Nontransversal Intersections | 129 |
| 6.1. Vector Fields with No Contours and No Homoclinic Trajectories | 129 |
| 6.2. A Theorem on Inaccessibility | 130 |
| 6.3. Moduli | 131 |
| 6.4. Systems with Contours | 132 |
| 6.5. Diffeomorphisms with Nontrivial Basic Sets | 133 |
| 6.6. Vector Fields in \mathbb{R}^3 with Trajectories Homoclinic to a Cycle .. | 133 |
| 6.7. Symbolic Dynamics | 134 |
| 6.8. Bifurcations of Smale Horseshoes | 136 |
| 6.9. Vector Fields on a Bifurcation Surface | 138 |
| 6.10. Diffeomorphisms with an Infinite Set of Stable Periodic Trajectories | 138 |
| § 7. Infinite Nonwandering Sets | 139 |
| 7.1. Vector Fields on the Two-Dimensional Torus | 139 |
| 7.2. Bifurcations of Systems with Two Homoclinic Curves of a Saddle | 140 |

| | |
|---|---------|
| 7.3. Systems with Feigenbaum Attractors | 142 |
| 7.4. Birth of Nonwandering Sets | 142 |
| 7.5. Persistence and Smoothness of Invariant Manifolds | 143 |
| 7.6. The Degenerate Family and Its Neighborhood in Function Space | 144 |
| 7.7. Birth of Tori in a Three-Dimensional Phase Space | 145 |
| § 8. Attractors and their Bifurcations | 145 |
| 8.1. The Likely Limit Set According to Milnor (1985) | 147 |
| 8.2. Statistical Limit Sets | 147 |
| 8.3. Internal Bifurcations and Crises of Attractors | 148 |
| 8.4. Internal Bifurcations and Crises of Equilibria and Cycles | 149 |
| 8.5. Bifurcations of the Two-Dimensional Torus | 150 |
| Chapter 4. Relaxation Oscillations | 154 |
| § 1. Fundamental Concepts | 155 |
| 1.1. An Example: van der Pol's Equation | 155 |
| 1.2. Fast and Slow Motions | 156 |
| 1.3. The Slow Surface and Slow Equations | 157 |
| 1.4. The Slow Motion as an Approximation to the Perturbed Motion | 158 |
| 1.5. The Phenomenon of Jumping | 159 |
| § 2. Singularities of the Fast and Slow Motions | 160 |
| 2.1. Singularities of Fast Motions at Jump Points of Systems with One Fast Variable | 160 |
| 2.2. Singularities of Projections of the Slow Surface | 161 |
| 2.3. The Slow Motion for Systems with One Slow Variable | 162 |
| 2.4. The Slow Motion for Systems with Two Slow Variables | 163 |
| 2.5. Normal Forms of Phase Curves of the Slow Motion | 164 |
| 2.6. Connection with the Theory of Implicit Differential Equations | 167 |
| 2.7. Degeneration of the Contact Structure | 168 |
| § 3. The Asymptotics of Relaxation Oscillations | 170 |
| 3.1. Degenerate Systems | 170 |
| 3.2. Systems of First Approximation | 171 |
| 3.3. Normalizations of Fast-Slow Systems with Two Slow Variables for $\varepsilon > 0$ | 173 |
| 3.4. Derivation of the Systems of First Approximation | 175 |
| 3.5. Investigation of the Systems of First Approximation | 175 |
| 3.6. Funnels | 177 |
| 3.7. Periodic Relaxation Oscillations in the Plane | 177 |
| § 4. Delayed Loss of Stability as a Pair of Eigenvalues Cross the Imaginary Axis | 179 |
| 4.1. Generic Systems | 179 |
| 4.2. Delayed Loss of Stability | 180 |

| | |
|---|-----|
| 4.3. Hard Loss of Stability in Analytic Systems of Type 2 | 181 |
| 4.4. Hysteresis | 181 |
| 4.5. The Mechanism of Delay | 182 |
| 4.6. Computation of the Moment of Jumping in Analytic Systems .. | 182 |
| 4.7. Delay Upon Loss of Stability by a Cycle | 185 |
| 4.8. Delayed Loss of Stability and "Ducks" | 185 |
| § 5. Duck Solutions | 185 |
| 5.1. An Example: A Singular Point on the Fold of the Slow Surface | 186 |
| 5.2. Existence of Duck Solutions | 188 |
| 5.3. The Evolution of Simple Degenerate Ducks | 189 |
| 5.4. A Semi-local Phenomenon: Ducks with Relaxation | 190 |
| 5.5. Ducks in \mathbb{R}^3 and \mathbb{R}^n | 191 |
| Recommended Literature | 193 |
| References | 195 |
| Additional References | 205 |

Preface

The word "bifurcation" means "splitting into two". "Bifurcation" is used to describe any sudden change that occurs while parameters are being smoothly varied in any system: dynamical, ecological, etc. Our survey is devoted to the bifurcations of phase portraits of differential equations – not only to bifurcations of equilibria and limit cycles, but also to perestroikas of the phase portraits of systems in the large and, above all, of their invariant sets and attractors. The statement of the problem in this form goes back to A.A. Andronov.

Connections with the theory of bifurcations penetrate all natural phenomena. The differential equations describing real physical systems always contain parameters whose exact values are, as a rule, unknown. If an equation modeling a physical system is structurally unstable, that is, if the behavior of its solutions may change qualitatively through arbitrarily small changes in its right-hand side, then it is necessary to understand which bifurcations of its phase portrait may occur through changes of the parameters.

Often model systems seem to be so complex that they do not admit meaningful investigation, above all because of the abundance of the variables which occur. In the study of such systems, some of the variables that change slowly in the course of the process described are, as a rule, assumed to be constant. The resulting system with a smaller number of variables can then be investigated. However, it is frequently impossible to consider the individual influences of the discarded terms in the original model. In this case, the discarded terms may be looked upon as typical perturbations, and, accordingly, the original model can be described by means of bifurcation theory applied to the reduced system.

Reformulating the well-known words of Poincaré on periodic solutions, one may say that bifurcations, like torches, light the way from well-understood dynamical systems to unstudied ones. L.D. Landau, and later E. Hopf, using this idea of bifurcation theory, offered a heuristic description of the transition from laminar to turbulent flow as the Reynolds number increases. In Landau's scenario this transition was accomplished through bifurcations of tori of steadily growing dimensions. Later on when the zoo of dynamical systems and their bifurcations had significantly grown, many papers appeared, describing – mainly at a physical level – the transition from regular (laminar) flow to chaotic (turbulent) flow. The chaotic behavior of the 3-dimensional model of Lorenz for convective motions has been explained with the aid of a chain of bifurcations. This explanation is not included in the present survey since, to save space, bifurcations of systems with symmetry have not been included. Lorenz's system is centrally symmetric.

The theory of relaxation oscillations, which deals with systems in which the parameters slowly change with time (these parameters are called slow variables), closely adjoins the theory of bifurcations in which parameters do not change with time. In "fast-slow" systems of relaxation oscillations, a slowness parameter

enters that characterizes the speed of change of the slow variables. When this parameter is zero, a fast-slow system transforms into a family studied in the theory of bifurcations, but at a nonzero value of the parameter specific phenomena arise which are sometimes called dynamical bifurcations.

In this survey, systematic use is made of the theory of singularities. The solutions to many problems of bifurcation theory (mostly of local ones) consist of presenting and investigating a so-called principal family – a kind of topological normal form for families of the class studied. The theory of singularities helps to guess at, and partially to investigate, principal families. This theory also describes the theory of bifurcations of equilibrium states, singularities of slow surfaces, slow motions in the theory of relaxation oscillations, etc.

We also note that finitely smooth normal forms of local families of differential equations are especially useful in the theory of nonlocal bifurcations. On one hand, these normal forms substantially simplify the presentation and investigation of bifurcations, and also simplify and clarify the proof and analysis of the results obtained. On the other hand, the nonlocal theory of bifurcations helps to select problems from the theory of normal forms that are important for applications. In our opinion, at the present time, the connection between the theory of normal forms and the nonlocal theory of bifurcations is not used often enough.

This survey includes, along with what is known, a series of new results, some of these are known to the authors through private communications. [Added in translation: The results mentioned below were new when the Russian text was written (1985). Now most of them have been published. The additional list of references is given after the main one and numbered.] Among these are eight new topics. The first is a complete investigation of bifurcations from equilibria in generic two-parameter families of vector fields on the plane with two intersecting invariant curves (the so-called reduced problem for two purely imaginary pairs, Sect. 4.5 and Sect. 4.6 of Chap. 1 (see Żoładek (1987))). The second is the construction of finitely smooth normal forms and functional moduli of the C^1 -classification of local families of vector fields and diffeomorphisms (Yu.S. Il'yashenko and S.Yu. Yakovenko, Sect. 5.7–5.10 of Chap. 2 (see Il'yashenko and Yakovenko [3*, 4*])). The third is the construction of a topological invariant of vector fields with a trajectory homoclinic to a saddle with complex eigenvalues (Sect. 5.6 of Chap. 3). The fourth is the description of a generic two-parameter deformation of a vector field with two homoclinic curves at a saddle, in which the bifurcation diagram of the deformation contains a continuum of components. (D.V. Turaev and L.P. Shil'nikov [9*], Sect. 7.2 of Chap. 3). The fifth result is the definition of a statistical limit set as a possible candidate for the concept of a physical attractor (Sect. 8.2 of Chap. 3 (Il'yashenko [2*])). The sixth one is the description of connections between the theory of implicit equations and relaxation oscillations, and the normalization of slow motions for fast-slow systems with one or two slow variables (see Arnol'd's theorem in Sect. 2.2–2.7 of Chap. 4 and the related paper by Davidov [1*]). The seventh result is normalization of fast-slow equations, and the explicit form and investigation of systems of first

approximation (Sect. 3.2–3.5 of Chap. 4; see the related paper by Teperin [8*]). The eighth and last one is the investigation of the delayed loss of stability in generic fast-slow systems as a pair of eigenvalues of a stable singular point of a fast equation crosses the imaginary axis (the birth of a cycle as a dynamical bifurcation (A.I. Nejshtadt, § 4 of Chap. 4); see [6*, 7*]). We also point here to a conjecture on the bifurcations in generic multiple parameter families of vector fields on the plane that is closely related to Hilbert's 16th problem (Sect. 2.8 of Chap. 3).

Our survey, inevitably, is incomplete. We did not include in it the comparatively few works on local bifurcations in three-parameter families and on nonlocal bifurcations in two-parameter families; some relevant citations are, however, given in the References. In describing nonlocal bifurcations we limited ourselves to only those things which happen on the boundary of the set of Morse-Smale systems. The theory of such bifurcations is substantially complete, although it is not very well known; it is mostly due to works of the Gor'kij school, which often have been published in sources that are hard to obtain. That part of the boundary of the set of Morse-Smale systems on which a countable set of nonwandering trajectories arise is not yet fully explored; but Sect. 7 of Chap. 3 is devoted to this problem. For reasons of consistency of style we often formulate known results in a form different from that in which they first appeared.

Chap. 1 and 2 were written by V.I. Arnol'd and Yu.S. Il'yashenko. Chap. 3, in its final version, was written by V.S. Afrajmovich and Yu.S. Il'yashenko with the participation of V.I. Arnol'd and L.P. Shil'nikov. Sect. 1.6 of Chap. 2 was written by V.S. Afrajmovich. Sects. 1 and 2 of Chap. 4 were written by V.I. Arnol'd, Sect. 3, except for Sect. 3.7, by Yu.S. Il'yashenko. Sect. 3.7 was written by N.Kh. Rozov, Sect. 4 by A.I. Nejshtadt, Sect. 5 by A.K. Zvonkin; the authors sincerely thank them. The authors do not claim that the list of References is complete. In its organization we followed the same principles as in the survey by Arnol'd and Il'yashenko (1985). The symbol \blacktriangle denotes the end of some formulations.