Nonlinear Systems Analysis

Second Edition

M. Vidyasagar

C • L • A • S • S • I • C • S

In Applied Mathematics

siam

42

Nonlinear Systems Analysis

Second Edition

M. Vidyasagar
Tata Consultancy Services
Hyderabad, India

siam
Society for Industrial and Applied Mathematics
Philadelphia

This SIAM edition is an unabridged republication of the work first published by Prentice-Hall, Inc., Englewood Cliffs, NJ, 1993.

10 9 8 7 6 5 4 3 2 1

Library of Congress Cataloging-in-Publication Data

Vidyasagar, M. (Mathukumalli), 1947-
Nonlinear systems analysis / M. Vidyasagar. -- 2nd ed.
p. cm. -- (Classics in applied mathematics ; 42)
Originally published: Englewood Cliffs, N.J. : Prentice Hall, c1993.
Includes bibliographical references and index.
ISBN 0-89871-526-1 (pbk.)
1. System analysis. 2. Differential equations, Nonlinear. I. Title. II. Series.

QA402 .V54 2002
515'.355--dc21 2002029214

Nonlinear Systems Analysis

SIAM's Classics in Applied Mathematics series consists of books that were previously allowed to go out of print. These books are republished by SIAM as a professional service because they continue to be important resources for mathematical scientists.

Classics in Applied Mathematics

C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences*

Johan G. F. Belinfante and Bernard Kolman, *A Survey of Lie Groups and Lie Algebras with Applications and Computational Methods*

James M. Ortega, *Numerical Analysis: A Second Course*

Anthony V. Fiacco and Garth P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*

F. H. Clarke, *Optimization and Nonsmooth Analysis*

George F. Carrier and Carl E. Pearson, *Ordinary Differential Equations*

Leo Breiman, *Probability*

R. Bellman and G. M. Wing, *An Introduction to Invariant Imbedding*

Abraham Berman and Robert J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*

Olvi L. Mangasarian, *Nonlinear Programming*

*Carl Friedrich Gauss, *Theory of the Combination of Observations Least Subject to Errors: Part One, Part Two, Supplement.* Translated by G. W. Stewart

Richard Bellman, *Introduction to Matrix Analysis*

U. M. Ascher, R. M. M. Mattheij, and R. D. Russell, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*

K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*

Charles L. Lawson and Richard J. Hanson, *Solving Least Squares Problems*

J. E. Dennis, Jr. and Robert B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*

Richard E. Barlow and Frank Proschan, *Mathematical Theory of Reliability*

*First time in print.

Classics in Applied Mathematics (continued)

Cornelius Lanczos, *Linear Differential Operators*

Richard Bellman, *Introduction to Matrix Analysis, Second Edition*

Beresford N. Parlett, *The Symmetric Eigenvalue Problem*

Richard Haberman, *Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow*

Peter W. M. John, *Statistical Design and Analysis of Experiments*

Tamer Başar and Geert Jan Olsder, *Dynamic Noncooperative Game Theory, Second Edition*

Emanuel Parzen, *Stochastic Processes*

Petar Kokotović, Hassan K. Khalil, and John O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*

Jean Dickinson Gibbons, Ingram Olkin, and Milton Sobel, *Selecting and Ordering Populations: A New Statistical Methodology*

James A. Murdock, *Perturbations: Theory and Methods*

Ivar Ekeland and Roger Témam, *Convex Analysis and Variational Problems*

Ivar Stakgold, *Boundary Value Problems of Mathematical Physics, Volumes I and II*

J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*

David Kinderlehrer and Guido Stampacchia, *An Introduction to Variational Inequalities and Their Applications*

F. Natterer, *The Mathematics of Computerized Tomography*

Avinash C. Kak and Malcolm Slaney, *Principles of Computerized Tomographic Imaging*

R. Wong, *Asymptotic Approximations of Integrals*

O. Axelsson and V. A. Barker, *Finite Element Solution of Boundary Value Problems: Theory and Computation*

David R. Brillinger, *Time Series: Data Analysis and Theory*

Joel N. Franklin, *Methods of Mathematical Economics: Linear and Nonlinear Programming, Fixed-Point Theorems*

Philip Hartman, *Ordinary Differential Equations, Second Edition*

Michael D. Intriligator, *Mathematical Optimization and Economic Theory*

Philippe G. Ciarlet, *The Finite Element Method for Elliptic Problems*

Jane K. Cullum and Ralph A. Willoughby, *Lanczos Algorithms for Large Symmetric Eigenvalue Computations, Vol. I: Theory*

M. Vidyasagar, *Nonlinear Systems Analysis, Second Edition*

To Charlie Desoer

आकाशात्पतितम् तोयम्
यथा गच्छति सागरम्
सर्वदेवनमस्कारः
केशवम् प्रति गच्छति

As all water falling from the sky
Eventually reaches the sea
So do salutations to various deities
Reach the same almighty

From *Sandhyavandanam* (A salute to the twilight)

లోకంబులు లోకేశులు
లోకస్థులు దెగిన తుది నలోకంబగు పెం
జీకటి కవ్వల నెవ్వం
డేకాకృతి వెలుగు నతని నే సేవింతున్

Beyond the worlds
Their rulers and their denizens
Beyond the unwordly void
The one Who shines alone
Him I worship

From *Andhra Maha Bhagavatam* by Bammera Potana
(c. 1400 A.D.)

PREFACE TO THE CLASSICS EDITION

I am delighted that SIAM has chosen to publish my book *Nonlinear Systems Analysis, Second Edition* in its Classics in Applied Mathematics series. Over the years, this series has republished several outstanding texts and monographs, and I feel flattered that my book has been included in this distinguished company.

When I wrote the first edition in 1978, the subject of nonlinear systems was something of a mystery to most control theorists. It was common to treat the analysis of nonlinear systems as a "bag of tricks." The first edition was among the first texts to treat the subject with some amount of rigor. Over the years, the control theory community came to accept a rigorous approach as the "natural" way to study nonlinear systems.

By the time I wrote the second edition in 1993, advances in the application of differential geometric methods to nonlinear analysis (and to some extent, synthesis) comprised an integral part of nonlinear systems analysis, as did the more traditional topics of Lyapunov stability and input-output stability. Accordingly, the second edition contains three extensive chapters, each devoted to one of these key topics. The chapters on Lyapunov and input-output stability are particularly valuable references because they contain material not found in many other nonlinear analysis texts.

In the nearly ten years since this second edition was originally published, nonlinear analysis has matured to a stage where every control theorist needs to possess a knowledge of its basic techniques. Moreover, the subject continues to be of interest not only to theorists but also to practitioners working in areas such as robotics, spacecraft control, motor speed control, and power systems.

I wish my readers many pleasant hours studying this timeless and ever-fascinating subject.

M. Vidyasagar
Hyderabad
April 2002

PREFACE

It is now more than a decade since I wrote the book *Nonlinear Systems Analysis*. Since that time, several developments have taken place in this area which have made it desirable to update the contents of the book. Accordingly, virtually the entire book has been rewritten. The most notable changes are the following:

1) During the past decade, there have been some significant advances in the area of nonlinear control system design based on the use of differential geometric methods. Thus it is imperative that anyone interested in nonlinear system theory should have at least a passing acquaintance with these methods. In this second edition, I have included a new chapter which discusses the differential geometric approach (Chapter 7). For ease of exposition, all systems are considered to evolve over an open subset of $\mathbf{R}^n$; thus the analysis is only local. Topics covered include reachability, observability, and feedback linearization (in both the input-state and input-output settings), zero dynamics, and the stabilization of linearizable systems. In addition to presenting the theory, I have also included some applications of the theory to problems in robotics. Motivated by this chapter, an interested and diligent student could pursue a more rigorous course of study with an advanced text.

2) Several significant results have been obtained in the "traditional" areas of Lyapunov stability and input-output stability since the writing of the first edition. Some of these results are included in the present edition, such as: observer-controller stabilization of nonlinear systems, and the stability of hierarchical systems (Section 5.8); relationships between Lyapunov stability and input-output stability (Section 6.3); and a useful class of transfer functions of distributed systems (Section 6.5). In addition to the above, Section 4.2, containing a rigorous analysis of the describing function method, is also new.

3) Various standard texts in stability theory have gone out of print, making their contents all but inaccessible to the student. Two examples of such books are: *Stability of Motion* by W. Hahn and *Feedback Systems: Input-Output Properties* by C. A. Desoer and myself. At the same time some of the techniques presented in these books are finding new and previously unsuspected applications. With this in mind, in the present edition I have included some relevant material from these and other classic books, such as the converse Lyapunov theory (Section 5.7), and the feedback stability of time-varying and/or nonlinear systems (Section 6.6).

4) In view of the increasing importance of digital computers, I have included a discussion of discrete-time systems in the chapters dealing with Lyapunov stability and input-output stability.

5) Three new appendices have been added. Appendix A describes a sixty year-old theorem due to Witold Orlicz, on the prevalence of differential equations with unique solutions. This paper is quite inaccessible, but its contents deserve wide dissemination. Appendix B gives a proof of the Kalman-Yacubovitch lemma, while Appendix C contains a proof of the Frobenius theorem. The contents of the last two appendices are of course readily available elsewhere, but their inclusion in the present text makes it more self-contained.

6) The original edition of this book contained examples which were mostly drill problems or exercises. During the recent years I have come to feel that nonlinear system theory is most useful in studying the behavior of an entire *class* of systems rather than a given *specific* system. Accordingly, several applications of nonlinear system theory have been included throughout the book. Most of them have to do with robotics in some form or other.

With these changes, the book is somewhat bigger than the first edition. It would be difficult to cover the entire book during a single semester. However, I hope its value as a reference has been enhanced by the changes. Chapter 2 contains basic material which should be covered in order to appreciate the remainder of the text. But a sincere attempt has been made to ensure that Chapters 3 through 7 are independent, so that an instructor can pick and choose material to suit his/her needs. Even within a chapter, it is possible to cover certain sections and omit others. A perusal of the Contents reveals the amount of flexibility available in putting together a suitable course from the contents of the text.

In spite of the enlargement in the size of the book, some topics which deserve the attention of system theorists are not included. Examples of such topics are chaotic motions, averaging analysis, Volterra series, bifurcation theory, and catastrophe theory. I have made a conscious decision to omit these topics, mainly to keep the length of the book within reasonable limits. But no study of nonlinear systems is complete without at least an introduction to these topics. Moreover, there are several excellent texts available addressing each of the above topics.

In the preface to the first edition, I wrote fancifully that the book could be used by "engineers, mathematicians, biologists *et cetera.*" Judging by the Science Citation Index, no biologists appear to have read the book (though two *social scientists* have, amazingly enough). More realistically, I would expect the present edition to be of interest primarily to engineers interested in a rigorous treatment of nonlinear systems, and to mathematicians interested in system theory. Though some aspects of control are covered in the book (especially in Chapter 7), the focus is still on analysis rather than synthesis. Hence I have retained the original title. I do expect that the book can be used not just in Electrical Engineering departments, but also in Mechanical Engineering departments, and perhaps in some departments of Applied Mathematics. Above all, I hope it will continue to serve as a reference source for standard results in nonlinear system analysis.

I would like to thank Toshiharu Sugie for his careful reading of early versions of Chapters 5 and 6. I would also like to thank those who reviewed the text, particularly Brian Anderson, Aristotle Araposthasis, Ragu Balakrishnan, Joseph Bentsman, Alan Desrochers, Brad Dickinson, Ashok Iyer, Bob Newcomb, Charles L. Phillips, and Irwin Sandberg.

It is my pleasure and honor to dedicate to this book to Professor Charles A. Desoer of the University of California at Berkeley. Though I was not privileged to be one of his Ph.D. students, I was fortunate enough to have come under his influence while still at a formative stage in my career. Any instances of originality, creativity and clarity in my research and exposition are but pale imitations of his shining example.

NOTE TO THE READER

All items within each section are numbered consecutively, be they equations, theorems, definitions, or something else. A reference such as "(17)" refers to the 17-th item *within the same section*. When it is necessary to refer to an item from another section, the full citation is given, e.g., "Theorem (5.1.16)." All theorems, lemmas, and definitions are stated in *italics*. In a definition, the concept being defined is displayed in **bold face.** The same convention is used in the running text as well. The use of italics in the running text is reserved for *emphasis*. The box symbol ■ is used to denote the end of a proof. In cases where there might be some ambiguity, the same symbol is also used to denote the end of an example. Lower-case bold letters such as $\mathbf{x}$ denote vectors, upper-case bold letters such as $\mathbf{A}$ denote matrices, and italic letters denote scalars; however, there are a few exceptions to this convention. For example, the identity matrix is denoted by I.

Finally, the reader is urged to attempt all the problems, since they are an integral part of the text. Happy reading!

CONTENTS

1. INTRODUCTION

The topic of this book is the analysis of nonlinear systems. The adjective "nonlinear" can be interpreted in one of two ways, namely: "*not* linear" or "not *necessarily* linear." The latter meaning is intended here.

Why should one study nonlinear systems? The fact is that virtually *all* physical systems are nonlinear in nature. Sometimes it is possible to describe the operation of a physical system by a linear model, such as a set of ordinary linear differential equations. This is the case, for example, if the mode of operation of the physical system does not deviate too much from the "nominal" set of operating conditions. Thus the analysis of linear systems occupies an important place in system theory. But in analyzing the behaviour of any physical system, one often encounters situations where the linearized model is inadequate or inaccurate; that is the time when the contents of this book may prove useful.

There are several important differences between linear systems and nonlinear systems: 1) In the case of linear systems described by a set of linear ordinary differential equations, it is often possible to derive *closed-form expressions* for the solutions of the system equations. In general, this is not possible in the case of nonlinear systems described by a set of nonlinear ordinary differential equations. As a consequence, it is desirable to be able to make some predictions about the behaviour of a nonlinear system even in the *absence* of closed-form expressions for the solutions of the system equations. This type of analysis, called **qualitative** analysis or **approximate** analysis, is much less relevant to linear systems. 2) The analysis of nonlinear systems makes use of a *wider variety* of approaches and mathematical tools than does the analysis of linear systems. The main reason for this variety is that no tool or methodology in nonlinear systems analysis is *universally* applicable (in a fruitful manner). Hence the nonlinear systems analyst needs a wide variety of tools in his or her arsenal. 3) In general, the level of mathematics needed to master the basic ideas of nonlinear systems analysis is higher than that for the linear case. Whereas matrix algebra usually occupies center stage in a first course in linear systems analysis, here we use ideas from more advanced topics such as functional analysis and differential geometry.

A commonly used model for a nonlinear system is

1 $$\dot{\mathbf{x}}(t) = \mathbf{f}[t, \mathbf{x}(t), \mathbf{u}(t)],\ \forall t \geq 0,$$

where t denotes time; $\mathbf{x}(t)$ denotes the value of the function $\mathbf{x}(\cdot)$ at time t and is an n-dimensional vector; $\mathbf{u}(t)$ is similarly defined and is an m-dimensional vector; and the function $\mathbf{f}$ associates, with each value of t, $\mathbf{x}(t)$, and $\mathbf{u}(t)$, a corresponding n-dimensional vector. Following common convention, this is denoted as: $t \in \mathbf{R}_+$, $\mathbf{x}(t) \in \mathbf{R}^n$, $\mathbf{u}(t) \in \mathbf{R}^m$, and $\mathbf{f}: \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^n$. Note that (1) is a first-order vector differential equation. The quantity $\mathbf{x}(t)$ is generally referred to as the **state** of the system at time t, while $\mathbf{u}(t)$ is called the **input** or the **control** function. It is clear that (1) represents a continuous-time system. Its discrete-time counterpart is

2 $$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k),\ k = 0, 1, 2, 3, \cdots,$$

which is a first-order vector difference equation. There is no loss of generality in assuming that the system at hand is described by a first-order (differential or difference) equation. To see this, suppose the system is described by the n-th order scalar differential equation

3 $$\frac{d^n y(t)}{dt^n} = h\left[t, y(t), \dot{y}(t), \cdots, \frac{d^{n-1} y(t)}{dt^{n-1}}, u(t)\right],\ \forall t \geq 0.$$

This equation can be recast in the form (1) by defining the n-dimensional state vector $\mathbf{x}(t)$ in the familiar way, namely

4 $$x_1(t) = y(t), x_2(t) = \dot{y}(t), \cdots, x_n(t) = \frac{d^{n-1} y(t)}{dt^{n-1}}.$$

Then (3) is equivalent to

5 $$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) \\ \dot{x}_n(t) &= h[t, x_1(t), x_2(t), \cdots, x_n(t), u(t)] \end{aligned}$$

Now (5) is of the form (1) with

6 $$\mathbf{x}(t) = [x_1(t) \cdots x_n(t)]',$$

7 $$\mathbf{f}(t, \mathbf{x}, u) = [x_1\ x_2\ \cdots x_n\ h(t, x_1, \cdots, x_n, u)]'.$$

More generally, even coupled nonlinear differential equations can be put into the form (1). Analogous remarks apply also to difference equations. In fact, much of the power of "modern" control theory derives from the generality and versatility of the state-space descriptions (1) and (2).

In studying the system (1), one can make a distinction between two aspects,[1] generally referred to as analysis and synthesis, respectively. Suppose the input function $\mathbf{u}(\cdot)$ in (1) is

[1] Henceforth attention is focused on the continuous-time system (1), with the understanding that all remarks apply, *mutatis mutandis*, to the discrete-time system (2).

specified (i.e., fixed), and one would like to study the behaviour of the corresponding function $\mathbf{x}(\cdot)$; this is usually referred to as **analysis**. Now suppose the problem is turned around: the system description (1) is given, as well as the desired behaviour of the function $\mathbf{x}(\cdot)$, and the problem is to find a suitable input function $\mathbf{u}(\cdot)$ that would cause $\mathbf{x}(\cdot)$ to behave in this desired fashion; this is usually referred to as **synthesis**. Most of this book is devoted to the analysis of nonlinear systems.

The rest of this chapter is devoted to introducing several commonly used terms. The system (1) is said to be **forced**, or to have an input; in contrast, a system described by an equation of the form

8 $\dot{\mathbf{x}}(t)=\mathbf{f}[t, \mathbf{x}(t)], \forall t \geq 0,$

is said to be **unforced.** Note that the distinction is not too precise. In the system (1), if $\mathbf{u}(\cdot)$ is specified, then it is possible to define a function $\mathbf{f_u}: \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ by

9 $\mathbf{f_u}(t, \mathbf{x})=\mathbf{f}[t, \mathbf{x}, \mathbf{u}(t)].$

In this case (1) becomes

10 $\dot{\mathbf{x}}(t)=\mathbf{f_u}[t, \mathbf{x}(t)], \forall t \geq 0.$

Moreover, if $\mathbf{u}(\cdot)$ is clear from the context, the subscript **u** on $\mathbf{f_u}$ is often omitted. In this case there is no distinction between (10) and (8). Thus it is safer to think of (8) as describing one of two possible cases: (i) there is no external input to the system, or (ii) there is an external input, which is kept fixed throughout the study.

11 **Definition** *The system (1) or (8) is said to be* **autonomous** *if the function* **f** *does not explicitly depend on its first argument t; it is said to be* **nonautonomous** *otherwise.*

Note that some authors use "time-invariant" instead of "autonomous" and "time-varying" instead of "nonautonomous."

Consider the system (1), and suppose it is autonomous, i.e., **f** is independent of t. Now suppose a *non-constant* input function $\mathbf{u}(\cdot)$ is applied. Then the corresponding function $\mathbf{f_u}$ defined in (10) may in fact depend on t [since $\mathbf{u}(t)$ depends on t]. The point to note is that a system may be either autonomous or nonautonomous depending on the context.

The next concept is central to nonlinear system theory.

12 **Definition** *A vector* $\mathbf{x}_0 \in \mathbf{R}^n$ *is said to be an* **equilibrium** *of the unforced system (8) if*

13 $\mathbf{f}(t, \mathbf{x}_0)=\mathbf{0}, \forall t \geq 0.$

If $\mathbf{x}_0$ is an equilibrium of the system (8), then the differential equation