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An Invitation to Algebraic Geometry

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Notes for the Second Printing

The second printing of this book corrects the many typos and errors that were brought to our attention by readers from around the world. We have also added a few exercises and clarified parts of the text. We are grateful to all the readers who have helped improve our book, but owe particular thanks to Brian Conrad, Sándor Kovács, Grisha Stewart, and especially to Rahim Zaare Nahandi of the University of Tehran, who is engaged in translating this volume into Persian.

Karen E. Smith
Berkeley, CA, USA
March 2003

Preface

These notes grew out of a course at the University of Jyväskylä in January 1996 as part of Finland's new graduate school in mathematics. The course was suggested by Professor Kari Astala, who asked me to give a series of ten two-hour lectures entitled "Algebraic Geometry for Analysts." The audience consisted mainly of two groups of mathematicians: Ph.D. students from the Universities of Jyväskylä and Helsinki, and mature mathematicians whose research and training were quite far removed from algebra. Finland has a rich tradition in classical and topological analysis, and it was primarily in this tradition that my audience was educated, although there were representatives of another well–known Finnish school, mathematical logic.

I tried to conduct a course that would be accessible to everyone, but that would take participants beyond the standard course in algebraic geometry. I wanted to convey a feeling for the underlying algebraic principles of algebraic geometry. But equally important, I wanted to explain some of algebraic geometry's major achievements in the twentieth century, as well as some of the problems that occupy its practitioners today. With such ambitious goals, it was necessary to omit many proofs and sacrifice some rigor.

In light of the background of the audience, few algebraic prerequisites were presumed beyond a basic course in linear algebra. On the other hand, the language of elementary point-set topology and some basic facts from complex analysis were used freely, as was a passing familiarity with the definition of a manifold.

My sketchy lectures were beautifully written up and massaged into this text by Lauri Kahanpää and Pekka Kekäläinen. This was a Herculean effort, no less because of the excellent figures Lauri created with the computer. Extensive revisions to the Finnish text were carried out together with Lauri and Pekka; later Will Traves joined in to help with substantial revisions to the English version. What finally resulted is this book, and it would not have been possible without the valuable contributions of all members of our four-author team.

This book is intended for the working or the aspiring mathematician who is unfamiliar with algebraic geometry but wishes to gain an appreciation of its foundations and its goals with a minimum of prerequisites. It is not intended to compete with such comprehensive introductions as Hartshorne's or Shafarevich's texts, to which we freely refer for proofs and rigor. Rather, we hope that at least some readers will be inspired to undertake more serious study of this beautiful subject. This book is, in short, An Invitation to Algebraic Geometry.

Karen E. Smith
Jyväskylä, Finland
August 1998

Acknowledgments

The notes of Ari Lehtonen, Jouni Parkkonen, and Tero Kilpeläinen complemented those of authors Lauri and Pekka in producing a typed version of the original lectures. Comments of Osmo Pekonen, Ari Lehtonen, and Lassi Kurittu then helped eradicate most of the misprints and misunderstandings marring the first draft, and remarks of Bill Fulton later helped improve the manuscript. Artistic advice from Virpi Kauko greatly improved the pictures, although we were able to execute her suggestions only with the help of Ari Lehtonen's prize-winning Mathematica skills. Computer support from Ari and from Bonnie Freidman at MIT made working together feasible in Jyväskylä and in the US despite different computer systems. The suggestions of Manuel Blickle, Mario Bonk, Bill Fulton, Juha Heinonen, Eero Hyry, and Irena Swanson improved the final exposition. We are especially grateful to Eero for comments on the Finnish version; as one of the few algebraic geometers working in Finland, he advised us on the choices we made regarding mathematical terminology in the Finnish language. The Finnish craftwork of Liisa Heinonen provided instructive props for the lectures, most notably the traditional Christmas Blowup, whose image appears inside the cover of this book. Cooperation with Ari Lehtonen was crucial in creating the photograph. The lectures were hosted by the University of Jyväskylä mathematics department, and we are indebted to the chairman, Tapani Kuusalo, for making them possible. Finally, author Karen acknowledges the patience of her daughter Sanelma during the final stages of work on this project, and the support of her husband and babysitter, Juha Heinonen.

Karen E. Smith
Ann Arbor, Michigan, USA
July 2000

Contents

1

Affine Algebraic Varieties

Algebraic geometers study zero loci of polynomials. More accurately, they study geometric objects, called algebraic varieties, that can be described locally as zero loci of polynomials. For example, every high school mathematics student has studied a bit of algebraic geometry, in learning the basic properties of conic sections such as parabolas and hyperbolas.

Algebraic geometry is a thriving discipline with a rich history. In ancient Greece, mathematicians such as Apollonius probably knew that a non-degenerate plane conic is uniquely determined by five tangent lines, a problem that would cause many modern students of algebraic geometry to pause. But it was not until the introduction of the Cartesian coordinate system in the seventeenth century, when it became possible to study conic sections by considering quadratic polynomials, that the subject of algebraic geometry could really take off.

By the mid-nineteenth century, algebraic geometry was flourishing. On the one hand, Riemann realized that compact Riemann surfaces can always be described by polynomial equations. On the other hand, particular examples of algebraic varieties, such as quadric and cubic surfaces (zero loci of a single quadratic or cubic polynomial in three variables) were well known and intensely studied. For example, it was understood that every quadric surface is perfectly covered by a family of disjoint lines, whereas every cubic surface contains exactly twenty-seven lines. Detailed studies of the ways in which these twenty-seven lines can be configured and how they can vary in families occupied the attention of numerous nineteenth-century mathematicians.

The remarkable intuition of the turn-of-the-century algebraic geometers eventually began to falter as the subject grew beyond its somewhat shaky logical foundations. Led by David Hilbert, mathematical culture shifted toward a greater emphasis on rigor, and soon algebraic geometry fell out of favor as gaps and even some errors appeared in the subject. Luckily, the spirit and techniques of algebraic geometry were kept alive, primarily by Italian mathematicians. By the mid-twentieth century, with the efforts of mathematicians such as David Hilbert and Emmy Noether, algebra was sufficiently developed so as to be able once again to support this beautiful and important subject.

In the middle of the twentieth century, Oscar Zariski and André Weil spent a good portion of their careers redeveloping the foundations of algebraic geometry on firm mathematical ground. This was not a mere process of filling in details left unstated before, but a revolutionary new approach, based on analyzing the algebraic properties of the set of all polynomial functions on an algebraic variety. These innovations revealed deep connections between previously separate areas of mathematics, such as number theory and the theory of Riemann surfaces, and eventually allowed Alexander Grothendieck to carry algebraic geometry to dizzying heights of abstraction in the last half of the century. This abstraction has simplified, unified, and greatly advanced the subject, and has provided powerful tools used to solve difficult problems. Today, algebraic geometry touches nearly every branch of mathematics.

An unfortunate effect of this late-twentieth-century abstraction is that it has sometimes made algebraic geometry appear impenetrable to outsiders. Nonetheless, as we hope to convey in this *Invitation to Algebraic Geometry*, the main objects of study in algebraic geometry, affine and projective algebraic varieties, and the main research questions about them, are as interesting and accessible as ever.

1.1 Definition and Examples

An algebraic variety is a geometric object that locally resembles the zero locus of a collection of polynomials. The idea of "locally resembling" is familiar to those who have studied manifolds, which are geometric objects locally resembling Euclidean space. We begin our study of algebraic geometry by considering this local picture in detail, the study of affine algebraic varieties.

Definition: An *affine algebraic variety* is the common zero set of a collection $\{F_i\}_{i\in I}$ of complex polynomials on complex n-space $\mathbb{C}^n$. We write

$$V = \mathbb{V}(\{F_i\}_{i\in I}) \subset \mathbb{C}^n$$

for this set of common zeros. Note that the indexing set I can be arbitrary, not necessarily finite or even countable.

For example, $V = \mathbb{V}(x_1, x_2) \subset \mathbb{C}^3$ is the complex line in $\mathbb{C}^3$ consisting of the x_3-axis.

This definition of an affine algebraic variety should be considered only a working preliminary definition. The problem is that it depends on considerations extrinsic to the objects themselves, namely the embedding of the affine variety in the particular affine space $\mathbb{C}^n$. Later, in Section 4.1, we will refine and expand our definition of an affine algebraic variety in order to make it a more intrinsic notion.

Strictly speaking, what we have defined above should be called a complex affine algebraic variety, because we are considering our varieties over the complex numbers. The field of complex numbers may be replaced by any other field, such as the field $\mathbb{R}$ of real numbers, the field $\mathbb{Q}$ of rational numbers, or even a finite field. For reasons we will see later, using complex numbers instead of real numbers makes algebraic geometry easier, and in order to keep this book as close as possible to familiar territory, we will work only over the complex numbers $\mathbb{C}$. However, the reader should bear in mind the possibility of using different fields; this flexibility allows algebraic geometry to be applied to problems in number theory (by using the rational numbers or some p-adic fields).

Examples:

(1) The space $\mathbb{C}^n$; the empty set; and one-point sets, *singletons*, are trivial examples of affine algebraic varieties:

$$\begin{aligned} \mathbb{C}^n &= \mathbb{V}(0); \\ \emptyset &= \mathbb{V}(1); \\ \{(a_1, \ldots, a_n)\} &= \mathbb{V}(x_1 - a_1, \ldots, x_n - a_n). \end{aligned}$$

We call the space $\mathbb{C}$ the *complex line*, and the space $\mathbb{C}^2$ the *complex plane.* Confusing as it may seem, the complex line $\mathbb{C}$ is called the "complex plane" in some other branches of mathematics. In general, the space $\mathbb{C}^n$ is called complex n-space or affine n-space.

When drawing a sketch of an affine algebraic variety V we will, of course, draw only its *real points* $V \cap \mathbb{R}^n$.

(2) An *affine plane curve* is the zero set of one complex polynomial in the complex plane $\mathbb{C}^2$. Figures 1.1, 1.2, and 1.3 show examples of plane curves.

(3) The zero set of a single polynomial in arbitrary dimension is called a *hypersurface* in $\mathbb{C}^n$. The quadratic cone in Figure 1.4 is a typical example of a hypersurface.

(4) The zero set of a linear (degree-one) polynomial is an affine algebraic variety called an affine *hyperplane.* For example, the line defined by $ax + by = c$ is a hyperplane in the complex plane $\mathbb{C}^2$, where here a, b, and c are complex scalars. A *linear affine algebraic variety* is the common zero set of

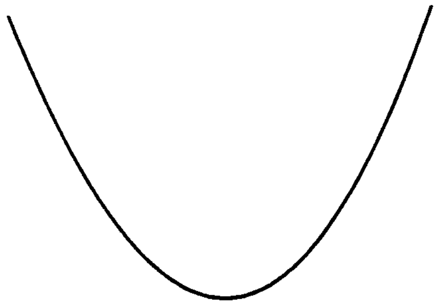

Figure 1.1. $\mathbb{V}(y - x^2) \subset \mathbb{C}^2$

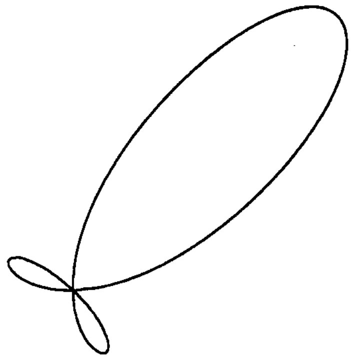

Figure 1.2. $\mathbb{V}(x^2y + xy^2 - x^4 - y^4) \subset \mathbb{C}^2$

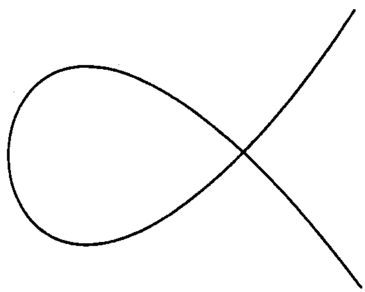

Figure 1.3. $\mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{C}^2$

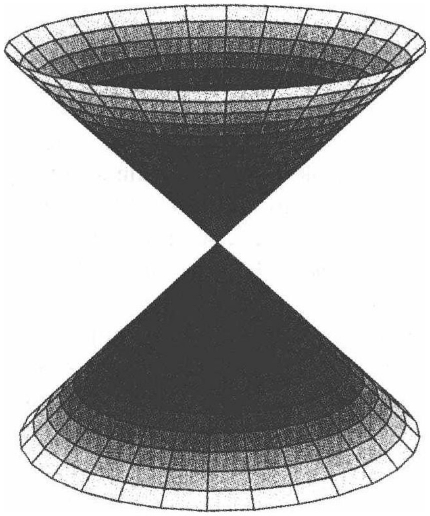

Figure 1.4. The quadratic cone $\mathbb{V}(x^2 + y^2 - z^2)$ in $\mathbb{C}^3$

a collection of linear polynomials of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n - b$$

in $\mathbb{C}^n$. If there are k linearly independent polynomials, the linear variety is a complex space of dimension $n - k$.

(5) The set of all $n \times n$ matrices can be identified with the set $\mathbb{C}^{n^2}$. This space contains some familiar objects as affine algebraic varieties. For instance, the subset $\mathbf{SL}(n, \mathbb{C})$ of matrices of determinant 1 forms an affine algebraic variety in $\mathbb{C}^{n^2}$, the hypersurface defined by the polynomial $\Delta - 1$, where Δ denotes the *determinant*

$$\Delta(x_{ij}) = \det \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix},$$

which is obviously a polynomial in the n^2 variables x_{ij}.

(6) A *determinantal variety* is the set in $\mathbb{C}^{n^2}$ of all matrices of rank at most k, where k is some fixed natural number. For $k \geq n$ the determinantal variety is the whole space $\mathbb{C}^{n^2}$, but for $k < n$ the rank of a matrix A is at most k if and only if all its $(k+1) \times (k+1)$ subdeterminants vanish. Because the subdeterminants are polynomials in the variables x_{ij}, the set of matrices of rank at most k is an affine algebraic variety.