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# WAVE PROPAGATION IN ELASTIC SOLIDS

BY

J. D. ACHENBACH

*The Technological Institute,  
Northwestern University, Evanston, Illinois*



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# WAVE PROPAGATION IN ELASTIC SOLIDS

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*To Marcia*

## PREFACE

The propagation of mechanical disturbances in solids is of interest in many branches of the physical sciences and engineering. This book aims to present an account of the theory of wave propagation in *elastic* solids. The material is arranged to present an exposition of the basic concepts of mechanical wave propagation within a one-dimensional setting and a discussion of formal aspects of elastodynamic theory in three dimensions, followed by chapters expounding on typical wave propagation phenomena, such as radiation, reflection, refraction, propagation in waveguides, and diffraction. The treatment necessarily involves considerable mathematical analysis. The pertinent mathematical techniques are, however, discussed at some length.

I hope that the book will serve a dual purpose. In addition to being a reference book for engineers and scientists in the broad sense, it is also intended to be a textbook for graduate courses in elastic wave propagation. As a text the book should be suitable for students who have completed first-year graduate courses in mechanics and mathematics. To add to its utility as a textbook each chapter is supplemented by a set of problems, which provide a useful test of the reader's understanding, as well as further illustrations of the basic ideas.

The book was developed from notes for a course offered to graduate students at Northwestern University. In the spring of 1969 a substantial part of the text was prepared in the form of typewritten notes for a series of lectures, while I was a visiting member of the faculty at the University of California in La Jolla. I am pleased to record my thanks for that opportunity. I also wish to express my gratitude to the Rector Magnificus of the Technological University of Delft and the Trustees of the Ir. Cornelis Gelderman Fund for inviting me to act as visiting professor in the department of mechanical engineering at the Technological University, in 1970-1971. While I was in Delft the larger part of the manuscript was completed. A sabbatical leave from Northwestern University during that period is gratefully acknowledged.

For the material of Chapters 5 and 6 I should like to acknowledge my indebtedness to the lectures and publications of Professor R. D. Mindlin. Substantial parts of Chapter 3 are based on the dissertation of Professor A. T. de Hoop, and on the work of Professor E. Sternberg. I am also indebted to many colleagues who read chapters of the book, and who provided me with their constructive criticism. Needless to say, I alone am responsible for errors of fact and logic.

A special word of thanks goes to Mrs. Ruth H. Meier who for many years provided excellent secretarial assistance, and who typed and retyped most of the manuscript as the material was arranged and rearranged.

Let me close with the wish that this book may convey some of the fascinating aspects of wave propagation as a phenomenon, and that it may have done justice to the elegance of the mathematical methods that have been employed.

J. D. A.



## TABLE OF CONTENTS

PREFACE	vii
INTRODUCTION	1
The propagation of mechanical disturbances	1
Continuum mechanics	3
Outline of contents	4
Historical sketch	7
Bibliography	8
 1. ONE-DIMENSIONAL MOTION OF AN ELASTIC CONTINUUM	 10
1.1. Introduction	10
1.2. Nonlinear continuum mechanics in one dimension	11
1.2.1. Motion	11
1.2.2. Deformation	11
1.2.3. Time-rates of change	12
1.2.4. Conservation of mass	14
1.2.5. Balance of momentum	15
1.2.6. Balance of energy	16
1.2.7. Linearized theory	17
1.2.8. Notation for the linearized theory	20
1.3. Half-space subjected to uniform surface tractions	21
1.4. Reflection and transmission	26
1.5. Waves in one-dimensional longitudinal stress	29
1.6. Harmonic waves	30
1.6.1. Traveling waves	30
1.6.2. Complex notation	32
1.6.3. Standing waves	32
1.6.4. Modes of free vibration	32
1.7. Flux of energy in time-harmonic waves	33
1.7.1. Time-average power per unit area	34
1.7.2. Velocity of energy flux	35
1.7.3. Energy transmission for standing waves	36
1.8. Fourier series and Fourier integrals	37
1.8.1. Fourier series	37
1.8.2. Fourier integrals	39
1.9. The use of Fourier integrals	41
1.10. Problems	42
 2. THE LINEARIZED THEORY OF ELASTICITY	 46
2.1. Introduction	46
2.2. Notation and mathematical preliminaries	47

2.2.1. Indicial notation	47
2.2.2. Vector operators	48
2.2.3. Gauss' theorem	49
2.2.4. Notation	50
2.3. Kinematics and dynamics	50
2.3.1. Deformation	50
2.3.2. Linear momentum and the stress tensor	51
2.3.3. Balance of moment of momentum	52
2.4. The homogeneous, isotropic, linearly elastic solid	52
2.4.1. Stress-strain relations	52
2.4.2. Stress and strain deviators	54
2.4.3. Strain energy	55
2.5. Problem statement in dynamic elasticity	55
2.6. One-dimensional problems	57
2.7. Two-dimensional problems	58
2.7.1. Antiplane shear	58
2.7.2. In-plane motions	58
2.8. The energy identity	59
2.9. Hamilton's principle	61
2.9.1. Statement of the principle	61
2.9.2. Variational equation of motion	63
2.9.3. Derivation of Hamilton's principle	64
2.10. Displacement potentials	65
2.11. Summary of equations in rectangular coordinates	66
2.12. Orthogonal curvilinear coordinates	68
2.13. Summary of equations in cylindrical coordinates	73
2.14. Summary of equations in spherical coordinates	75
2.15. The ideal fluid	78
3. ELASTODYNAMIC THEORY	79
3.1. Introduction	79
3.2. Uniqueness of solution	80
3.3. The dynamic reciprocal identity	82
3.4. Scalar and vector potentials for the displacement field	85
3.4.1. Displacement representation	85
3.4.2. Completeness theorem	85
3.5. The Helmholtz decomposition of a vector	88
3.6. Wave motion generated by body forces	89
3.6.1. Radiation	89
3.6.2. Elastodynamic solution	93
3.7. Radiation in two dimensions	93
3.8. The basic singular solution of elastodynamics	96
3.8.1. Point load	96
3.8.2. Center of compression	101
3.9. Three-dimensional integral representation	102
3.9.1. Kirchhoff's formula	103
3.9.2. Elastodynamic representation theorem	104
3.10. Two-dimensional integral representations	105
3.10.1. Basic singular solutions	106
3.10.2. Antiplane line load	107
3.10.3. In-plane line load	108
3.10.4. Integral representations	109

3.11. Boundary-value problems	110
3.12. Steady-state time-harmonic response	115
3.12.1. Time-harmonic source	115
3.12.2. Helmholtz's equation	116
3.12.3. Helmholtz's first (interior) formula	117
3.12.4. Helmholtz's second (exterior) formula	117
3.12.5. Steady-state solutions in two dimensions	118
3.13. Problems	119
 4. ELASTIC WAVES IN AN UNBOUNDED MEDIUM	 122
4.1. Plane waves	122
4.2. Time-harmonic plane waves	124
4.2.1. Inhomogeneous plane waves	125
4.2.2. Slowness diagrams	127
4.3. Wave motions with polar symmetry	128
4.3.1. Governing equations	128
4.3.2. Pressurization of a spherical cavity	129
4.3.3. Superposition of harmonic waves	132
4.4. Two-dimensional wave motions with axial symmetry	135
4.4.1. Governing equations	135
4.4.2. Harmonic waves	136
4.5. Propagation of wavefronts	138
4.5.1. Propagating discontinuities	138
4.5.2. Dynamical conditions at the wavefront	140
4.5.3. Kinematical conditions at the wavefront	141
4.5.4. Wavefronts and rays	142
4.6. Expansions behind the wavefront	144
4.7. Axial shear waves by the method of characteristics	148
4.8. Radial motions	152
4.9. Homogeneous solutions of the wave equation	154
4.9.1. Chaplygin's transformation	154
4.9.2. Line load	156
4.9.3. Shear waves in an elastic wedge	157
4.10. Problems	160
 5. PLANE HARMONIC WAVES IN ELASTIC HALF-SPACES	 165
5.1. Reflection and refraction at a plane interface	165
5.2. Plane harmonic waves	166
5.3. Flux of energy in time-harmonic waves	166
5.4. Joined half-spaces	168
5.5. Reflection of SH-waves	170
5.6. Reflection of P-waves	172
5.7. Reflection of SV-waves	177
5.8. Reflection and partition of energy at a free surface	181
5.9. Reflection and refraction of SH-waves	182
5.10. Reflection and refraction of P-waves	185
5.11. Rayleigh surface waves	187
5.12. Stoneley waves	194
5.13. Slowness diagrams	196
5.14. Problems	198

6. HARMONIC WAVES IN WAVEGUIDES	202
6.1. Introduction	202
6.2. Horizontally polarized shear waves in an elastic layer	203
6.3. The frequency spectrum of SH-modes	206
6.4. Energy transport by SH-waves in a layer	208
6.5. Energy propagation velocity and group velocity	211
6.6. Love waves	218
6.7. Waves in plane strain in an elastic layer	220
6.8. The Rayleigh-Lamb frequency spectrum	226
6.9. Waves in a rod of circular cross section	236
6.10. The frequency spectrum of the circular rod of solid cross section	240
6.10.1. Torsional waves	241
6.10.2. Longitudinal waves	242
6.10.3. Flexural waves	246
6.11. Approximate theories for rods	249
6.11.1. Extensional motions	250
6.11.2. Torsional motions	251
6.11.3. Flexural motions – Bernoulli-Euler model	251
6.11.4. Flexural motions – Timoshenko model	252
6.12. Approximate theories for plates	254
6.12.1. Flexural motions – classical theory	255
6.12.2. Effects of transverse shear and rotary inertia	256
6.12.3. Extensional motions	257
6.13. Problems	258
7. FORCED MOTIONS OF A HALF-SPACE	262
7.1. Integral transform techniques	262
7.2. Exponential transforms	264
7.2.1. Exponential Fourier transform	265
7.2.2. Two-sided Laplace transform	267
7.2.3. One-sided Laplace transform	268
7.3. Other integral transforms	269
7.3.1. Fourier sine transform	270
7.3.2. Fourier cosine transform	270
7.3.3. Hankel transform	270
7.3.4. Mellin transform	271
7.4. Asymptotic expansions of integrals	271
7.4.1. General considerations	271
7.4.2. Watson's lemma	272
7.4.3. Fourier integrals	273
7.4.4. The saddle point method	273
7.5. The methods of stationary phase and steepest descent	274
7.5.1. Stationary-phase approximation	274
7.5.2. Steepest-descent approximation	278
7.6. Half-space subjected to antiplane surface disturbances	283
7.6.1. Exact solution	284
7.6.2. Asymptotic representation	288
7.6.3. Steepest-descent approximation	288
7.7. Lamb's problem for a time-harmonic line load	289
7.7.1. Equations governing a state of plane strain	290
7.7.2. Steady-state solution	291
7.8. Suddenly applied line load in an unbounded medium	295

7.9. The Cagniard-de Hoop method	298
7.10. Some observations on the solution for the line load	301
7.11. Transient waves in a half-space	303
7.12. Normal point load on a half-space	310
7.12.1. Method of solution	310
7.12.2. Normal displacement at $z = 0$	313
7.12.3. Special case $\lambda = \mu$	316
7.13. Surface waves generated by a normal point load	318
7.14. Problems	321
 8. TRANSIENT WAVES IN LAYERS AND RODS	 326
8.1. General considerations	326
8.2. Forced shear motions of a layer	327
8.2.1. Steady-state harmonic motions	328
8.2.2. Transient motions	330
8.3. Transient in-plane motion of a layer	331
8.3.1. Method of solution	332
8.3.2. Inversion of the transforms	335
8.3.3. Application of the method of stationary phase	337
8.4. The point load on a layer	342
8.5. Impact of a rod	344
8.5.1. Exact formulation	347
8.5.2. Inversion of the transforms	349
8.5.3. Evaluation of the particle velocity for large time	350
8.6. Problems	353
 9. DIFFRACTION OF WAVES BY A SLIT	 357
9.1. Mixed boundary-value problems	357
9.2. Antiplane shear motions	358
9.2.1. Green's function	359
9.2.2. The mixed boundary-value problem	362
9.3. The Wiener-Hopf technique	365
9.4. The decomposition of a function	369
9.4.1. General procedure	369
9.4.2. Example: the Rayleigh function	371
9.5. Diffraction of a horizontally polarized shear wave	372
9.6. Diffraction of a longitudinal wave	380
9.6.1. Formulation	380
9.6.2. Application of the Wiener-Hopf technique	382
9.6.3. Inversion of transforms	385
9.7. Problems	388
 10. THERMAL AND VISCOELASTIC EFFECTS, AND EFFECTS OF ANISOTROPY AND NON- LINEARITY	 391
10.1. Thermal effects	391
10.2. Coupled thermoelastic theory	392
10.2.1. Time-harmonic plane waves	392
10.2.2. Transverse waves	394
10.2.3. Longitudinal waves	394
10.2.4. Transient waves	396
10.2.5. Second sound	398
10.3. Uncoupled thermoelastic theory	399

10.4. The linearly viscoelastic solid	399
10.4.1. Viscoelastic behavior	399
10.4.2. Constitutive equations in three dimensions	401
10.4.3. Complex modulus	402
10.5. Waves in viscoelastic solids	403
10.5.1. Time-harmonic waves	403
10.5.2. Longitudinal waves	403
10.5.3. Transverse waves	404
10.5.4. Transient waves	404
10.5.5. Propagation of discontinuities	407
10.6. Waves in anisotropic materials	409
10.7. A problem of transient nonlinear wave propagation	412
10.8. Problems	417
 AUTHOR INDEX	 420
SUBJECT INDEX	422

## INTRODUCTION

### **The propagation of mechanical disturbances**

The local excitation of a medium is not instantaneously detected at positions that are at a distance from the region of excitation. It takes time for a disturbance to propagate from its source to other positions. This phenomenon of propagation of disturbances is well known from physical experience, and some illustrative examples immediately come to mind. Thus an earthquake or an underground nuclear explosion is recorded in another continent well after it has occurred. The report of a distant gun is heard after the projectile has arrived, because the velocity of disturbances in air, i.e., the speed of sound, is generally smaller than the velocity of the projectile. More familiar manifestations of the propagation of disturbances are waves in a rope or propagating ripples on the surface of water. These examples illustrate mechanical wave motions or mechanical wave propagation.

Mechanical waves originate in the forced motion of a portion of a deformable medium. As elements of the medium are deformed the disturbance is transmitted from one point to the next and the disturbance, or wave, progresses through the medium. In this process the resistance offered to deformation by the consistency of the medium, as well as the resistance to motion offered by inertia, must be overcome. As the disturbance propagates through the medium it carries along amounts of energy in the forms of kinetic and potential energies. Energy can be transmitted over considerable distances by wave motion. The transmission of energy is effected because motion is passed on from one particle to the next and not by any sustained bulk motion of the entire medium. Mechanical waves are characterized by the transport of energy through motions of particles about an equilibrium position. Thus, bulk motions of a medium such as occur, for example, in turbulence in a fluid are not wave motions.

Deformability and inertia are essential properties of a medium for the transmission of mechanical wave motions. If the medium were not deformable any part of the medium would immediately experience a disturbance in the

form of an internal force or an acceleration upon application of a localized excitation. Similarly, if a hypothetical medium were without inertia there would be no delay in the displacement of particles and the transmission of the disturbance from particle to particle would be effected instantaneously to the most distant particle. Indeed, in later chapters it will be shown analytically that the velocity of propagation of a mechanical disturbance always assumes the form of the square root of the ratio of a parameter defining the resistance to deformation and a parameter defining the inertia of the medium. All real materials are of course deformable and possess mass and thus all real materials transmit mechanical waves.

The inertia of a medium first offers resistance to motion, but once the medium is in motion inertia in conjunction with the resilience of the medium tends to sustain the motion. If, after a certain interval the externally applied excitation becomes stationary, the motion of the medium will eventually subside due to frictional lossess and a state of static deformation will be reached. The importance of dynamic effects depends on the relative magnitudes of two characteristic times: the time characterizing the external application of the disturbance and the characteristic time of transmission of disturbances across the body.

Suppose we consider a solid body subjected to an external disturbance  $F(t)$  applied at a point  $P$ . The purpose of an analysis is to compute the deformation and the distribution of stresses as functions of the spatial coordinates and time. If the greatest velocity of propagation of disturbances is  $c$ , and if the external disturbance is applied at time  $t = 0$ , the disturbed regions at times  $t = t_1$  and  $t = t_2$  are bounded by spheres centered at the point  $P$ , with radii  $ct_1$  and  $ct_2$ , respectively. Thus the whole of the body is disturbed at time  $t = r/c$ , where  $r$  is the largest distance within the body measured from the point  $P$ . Now suppose that the significant changes in  $F(t)$  take place over a time  $t_a$ . It can then be stated that dynamic effects are of importance if  $t_a$  and  $r/c$  are of the same order of magnitude. If  $t_a \gg r/c$ , the problem is quasistatic rather than dynamic in nature and inertia effects can be neglected. Thus for bodies of small dimensions a wave propagation analysis is called for if  $t_a$  is small. If the excitation source is removed the body returns to rest after a certain time. For excitation sources that are applied and removed, the effects of wave motion are important if the time interval of application is of the same order of magnitude as a characteristic time of transmission of a disturbance across the body. For bodies of finite dimensions this is the case for loads of explosive origins or for impact loads. For sustained external disturbances the effects of wave motions need be considered if the externally



applied disturbances are rapidly changing with time, i.e., if the frequency is high.

In mathematical terms a traveling wave in one dimension is defined by an expression of the type  $f = f(x - ct)$ , where  $f$  as a function of the spatial coordinate  $x$  and the time  $t$  represents a disturbance in the values of some physical quantity. For mechanical waves  $f$  generally denotes a displacement, a particle velocity or a stress component. The function  $f(x - ct)$  is called a simple wave function, and the argument  $x - ct$  is the phase of the wave function. If  $t$  is increased by any value, say  $\Delta t$ , and simultaneously  $x$  is increased by  $c\Delta t$ , the value of  $f(x - ct)$  is clearly not altered. The function  $f(x - ct)$  thus represents a disturbance advancing in the positive  $x$ -direction with a velocity  $c$ . The velocity  $c$  is termed the phase velocity. The propagating disturbance represented by  $f(x - ct)$  is a special wave in that the shape of the disturbance is unaltered as it propagates through the medium.

### Continuum mechanics

Problems of the motion and deformation of substances are rendered amenable to mathematical analysis by introducing the concept of a continuum or continuous medium. In this idealization it is assumed that properties averaged over a very small element, for example, the mean mass density, the mean displacement, the mean interaction force, etc., vary continuously with position in the medium, so that we may speak about *the* mass density, *the* displacement and *the* stress, as functions of position and time. Although it might seem that the microscopic structure of real materials is not consistent with the concept of a continuum, the idealization produces very useful results, simply because the lengths characterizing the microscopic structure of most materials are generally much smaller than any lengths arising in the deformation of the medium. Even if in certain special cases the microstructure gives rise to significant phenomena, these can be taken into account within the framework of the continuum theory by appropriate generalizations.

The analysis of disturbances in a medium within the context of the continuum concept belongs to the time-tested discipline of continuum mechanics. In achieving the traditional objective of determining the motion and deformation generated by external excitations the analysis passes through two major stages. In the first stage the body is idealized as a continuous medium and the physical phenomena are described in mathematical terms by introducing appropriate mathematical abstractions. Completion of this