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Stability and Stable Oscillations in Discrete Time Systems

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and
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Stability and Stable Oscillations in Discrete Time Systems

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A series edited by ***Saber Elaydi,*** *Trinity University, San Antonio, Texas, USA* and ***Gerry Ladas,*** *University of Rhode Island, Kingston, USA*

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Stability and Stable Oscillations in Discrete Time Systems
Aristide Halanay and Vladimir Răsvan

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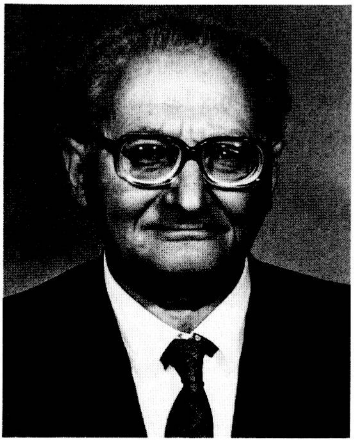

Professor Aristide Halanay
(1924–1997)

The distinguished Romanian mathematician Professor Aristide Halanay, one of the founders of the modern Romanian school of ordinary differential equations, was born in the small Romanian town Ramnicu-Sarat. In October 1946 he was appointed as an instructor in algebra at Bucharest University where he later obtained a Diploma in Mathematics in 1947. From 1949–1952, Aristide was a graduate student at the "Lomonosov" State University in Moscow under the supervision of Professor V. V. Nemytskii, one of the pioneers of the qualitative theory of ordinary differential equations. In 1952 Halanay became a Candidate in Physico-Mathematical Sciences (the Russian equivalent of a PhD) with a thesis on 2nd order linear differential equations with almost periodic coefficients. He was an associate professor from 1953–1968 and professor from 1968–1989 on the Faculty of Mathematics at Bucharest University.

Aristide was a member of the editorial board of the *Journal of Differential Equations* from its inception in 1965. His scientific activity started in 1947 with papers in algebra, and saw immense growth exemplified by production of approximately 200 papers and 12 books and monographs. He closely followed new trends in mathematical research and, over the years, made important contributions to the foundation of the theory of differential equations with delayed argument; stability and oscillations; singular perturbations; absolute stability of control systems; qualitative theory of discrete and stochastic systems; and optimal control of delay and discrete systems. In the last quarter of the twentieth century, Halanay showed an almost exclusive interest in problems that are most directly connected with practical applications, for example, stability of synchronous machines, dynamics of hydropower plants, and so on. The relevance and importance of Halanay's results are perhaps best illustrated by the fact that his monograph *Differential Equations, Stability, Oscillations, Time Lags*, first published in Romanian in 1963, was later published in English and Japanese and is one of the most cited references in the field.

Whether teaching or doing scientific research, he showed tremendous energy in everything he did. Amazingly creative and a hard worker, Aristide worked on mathematics until his last days. To those who knew and cherished him, he will be greatly missed.

SERIES EDITORS' PREFACE

This is a new volume in the book series *Advances in Discrete Mathematics and Applications*. The series will be a forum for all aspects of discrete mathematics and will act as a unifying force in the field, presenting books in areas such as numerical analysis, discrete dynamical systems, chaos theory, fractals, game theory, stability, control theory, complex dynamics, computational linear algebra, boundary value problems, oscillation theory, asymptotic theory, orthogonal polynomials, special functions, combinatorics and functional equations. Volumes on applications of difference equations in science and engineering will also be considered for publication.

Advances in Discrete Mathematics and Applications will publish textbooks for both the upper undergraduate and graduate levels. In addition, it will publish advanced works at the research level.

We hope to meet the growing needs of the mathematical community for books in discrete mathematics.

Saber Elaydi

Gerry Ladas

PREFACE

This monograph was written as a result of the establishment of the book series devoted to discrete dynamical systems by Professors Elaydi and Ladas. They kindly invited us to participate in this project.

It was natural for us to use this opportunity to review and reevaluate our experience in the research on discrete time systems. For many years we worked jointly on several problems in stability and oscillations, both in continuous time and discrete time, combining the experience of a professional mathematician and a theoretical engineer.

In this monograph we have focussed on regular behavior related to stability and stable oscillations. While writing this book, we were cognizant of the large volume of recent literature on this subject, including contributions from Professors Elaydi and Ladas. However, it is almost impossible to thoroughly cover all areas of the subject through such contributions. By basing the book essentially on our research experience, it is our aim to present a unique perspective of this subject.

We take this opportunity to again thank Professors Elaydi and Ladas for connecting us with the series. Thanks also go to Professor Anton Batatorescu for his excellent performance in processing the manuscript and the publisher for their effective cooperation.

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CHAPTER 1

INTRODUCTION

Discrete-time system dynamics is a topic of broad interest; the main reason for this interest comes from the variety of the sources of discrete-time dynamical models. We may cite:

1^0 Discrete-time models determined by the nature of the described processes: this is particularly true for economics, biology, physiology and discrete-time information processing.

2^0 Discrete-time models induced by the impulses occurring in continuous-time systems.

3^0 Discrete models occurring in controlled systems when the feedback information used in control generation is composed of output samples obtained through sampling intervals of time.

4^0 Discrete systems occurring during numerical treatment of continuous time systems.

But the interest in discrete-time systems may also be explained by the simplicity of their treatment - it requires minimal computational and graphical resources to obtain the solutions of the associated difference equations and follow this behavior. Since difference equations may be viewed as recurrence relations, their treatment seems much simpler that the one of differential equations.

In many cases the models describing biological or economic processes lead to what is now called *chaotic behavior* - a rather complicated, irregular picture of the evolution trajectories. In fact one of the sources for the widespread popularity of the discrete models lies in the fact that the simplest nonlinear ones may display such chaotic behavior that some people - mainly physicists - consider them to be some type of paradigm that may enhance the various phenomena of the physical universe.

In opposition to such behavior, the other three sources we mentioned for discrete-time systems display mainly regular behavior, stability. This fact is less obvious in the case of the systems with shocks where the impulses may generate chaotic behavior. Nevertheless, in this case also of interest are mainly those cases when the impulses, controlled or not, generate stable processes. The stability problems are especially important in the case of sampled data systems and also when the discrete-time system corresponds to a computational process.

In the following we shall illustrate these aspects using some simple examples.

1.1. MODELS WITH DISCRETE STORAGE OF THE INFORMATION

In several cases discrete-time equations are obtained when the physical parameter of interest is stored through intervals of time ("from time to time"). Following Maynard-Smith (1974), May (1995), Elaydi (1996), Kocic and Ladas (1993) let us cite the following discrete-time models in biology:

- the self-limited growth model (the logistic equation):

$$x_{t+1} = ax_t\left(1 - x_t\right)\ ,\ t = 0, 1, 2, ... \tag{1.1}$$

- the discrete-generation predator-prey model (Maynard-Smith (1974)):

$$\begin{aligned} x_{t+1} &= ax_t - bx_t y_t \\ y_{t+1} &= -cy_t + dx_t y_t \end{aligned} \tag{1.2}$$

- the host-parasitoid interaction model (May (1995)):

$$\begin{aligned} H_{t+1} &= R_0 H_t F\left(P_t, H_t\right) \\ P_{t+1} &= cH_t\left[1 - F\left(P_t, H_t\right)\right] \end{aligned} \tag{1.3}$$

  where H_t and P_t represent the number (or density) of hosts and parasites in generation t.
- the Leslie model of the age structure for a population (Svirežev and Logofet (1978)):

$$\begin{aligned} x_{t+1}^1 &= \sum_1^n b_i x_t^i\ ,\ b_i \geq 0 \\ x_{t+1}^{i+1} &= s_i x_t^i\ ,\ 0 < s_i \leq 1\ ,\ i = \overline{1, n-1} \end{aligned} \tag{1.4}$$

- the Kermack-McKendrick model of epidemics (Kocic and Ladas (1993), p. 195):

$$\begin{aligned} S_{t+1} &= e^{-\alpha I_t} S_t \\ I_{t+1} &= \beta I_t + \left(1 - e^{-\alpha I_t}\right) S_t \\ R_{t+1} &= (1-\beta) I_t + R_t \end{aligned} \tag{1.5}$$

- discrete competitive systems (Kocic and Ladas (1993), p. 199):

$$\begin{aligned} x_{t+1} &= x_t f\left(a x_t + b y_t\right) \\ y_{t+1} &= y_t g\left(c x_t + d y_t\right), \end{aligned} \tag{1.6}$$

where a, b, c, d are positive constants, f, g are decreasing on $(0, \infty)$ and there exist $\bar{x} > 0$, $\bar{y} > 0$ such that $f(\bar{x}) = g(\bar{y}) = 1$.

In order to illustrate the dynamic behavior for systems described by such models *we shall consider the logistic equation in some detail.* It is quite well known (Elaydi (1996)) that while elementary in structure, this equation displays various types of behavior, from very simple to very complicated ones.

A. We shall call solution of (1.1) a sequence $\{x_t\}_t$ defined recurrently by:

$$x_{t+1} = a x_t (1 - x_t)$$

with given x_0. In fact x_t is the t-iterate of the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax(1-x)$. Remark that $f(0) = f(1) = 0$ and $\max f(x) = f(1/2) = a/4$. It follows that $f(x)$ maps the interval $[0,1]$ into itself provided $0 < a \le 4$. Since in population dynamics x_t is a rated population density, we shall have for physical reasons $0 \le x_t \le 1$. From the mathematical point of view this means that the model may be considered correct if $x_0 \in [0,1]$ implies $x_t \in [0,1]$ for all positive integers t; we shall say *that* $[0,1]$ *is an invariant set for* (1.1). The above considerations show that this is true for $a \in (0,4]$.

In the following we shall assume $a \in (0,4]$ and consider equation (1.1) only on the interval $[0,1]$. *The constant solutions of* (1.1) *are called equilibria*; they are given by

$$x = f(x) = ax(1-x)$$

being the fixed points of the mapping $f(x) : \hat{x}^1 = 0$, $\hat{x}^2 = (a-1)/a$. Remark that $0 \le \hat{x}^2 < 1$ only if $a \ge 1$ otherwise this equilibrium is not contained in the considered interval.

Let $0 < a < 1$. On the interval $[0, 1]$ we shall have

$$x_{t+1} - x_t = x_t \left[a\left(1 - x_t\right) - 1\right] < 0\,;$$

hence any sequence x_t with $0 \leq x_0 \leq 1$ is decreasing. Since $x_t \geq 0$ we shall have $\lim\limits_{t\to\infty} x_t \geq 0$; from the existence of this limit we deduce that letting $t \to \infty$ in (1.1) is legitimate, the limit being a nonnegative equilibrium, i.e., $\hat{x}^1 = 0$ since $\hat{x}^2 < 0$ in this case. We obtained that the equilibrium $\hat{x}^1 = 0$ is attractive for the solutions starting in $[0, 1]$.

Let now $a > 1$; in this case both equilibria belong to the interval $[0, 1] : 0 = \hat{x}^1 < (a-1)/a = \hat{x}^2 < 1$.

Consider the deviations with respect to the nonzero equilibrium $\hat{x}^2$ namely $z_t = x_t - \hat{x}^2$; we shall have

$$z_{t+1} = (2-a)\, z_t - a z_t^2 \tag{1.7}$$

If $1 < a < 3$ then $|2 - a| < 1$ and for z_0 small enough we shall have $z_t \to 0$ exponentially by the Liapunov theorem on stability by the first approximation (see Section 2.3 of the book). Since the deviations from the equilibrium $\hat{x}^2$ tend exponentially to zero (for $t \to \infty$) we shall say that $\hat{x}^2 = (a-1)/a$ is an *attractor*. Under the condition $1 < a < 3$ the other equilibrium, $\hat{x}^1 = 0$ is *repulsive*. Indeed let $x_0 > 0$ be a neighborhood of $\hat{x}^1 = 0$ e.g. satisfying $x_0 < (a-1)/a$. We have also

$$x_{t+1} - x_t = x_t\left(a - 1 - a x_t\right) > 0\,,$$

provided $a x_t < a - 1$. It follows that any solution sequence x_t that satisfies $0 < x_0 < (a-1)/a$ is strictly increasing, i.e., the deviations from $\hat{x}^1$ even if small initially are increasing which shows repulsiveness of this equilibrium in the considered case.

Let now $a > 3$. In this case the equilibrium $\hat{x}^2$ is also repulsive, as follows from the equation in deviations (1.7); indeed we shall have

$$\begin{aligned} z_{t+1}^2 - z_t^2 &= \left[(2-a)\, z_t - a z_t^2\right]^2 - z_t^2 = \\ &= z_t^2 \left(a - 1 + a z_t\right)\left(a - 3 + a z_t\right) = \\ &= (a-3)^2 z_t^2 + z_t^2 \left[2\,(a-3) + 2a\,(a-2)\, z_t + a z_t^2\right] \end{aligned}$$

The polynomial $a\lambda^2 + 2a\,(a-2)\,\lambda + 2\,(a-3)$ has negative roots for $a > 3$ (its coefficients are strictly positive), hence we may take some $\delta > 0$ sufficiently small in order that for $|\lambda| < \delta$ this polynomial is positive. As long as $|z_t| < \delta$ we have

$$z_{t+1}^2 - z_t^2 > (a-3)^2 z_t^2$$

which shows that $\{z_t\}$ is an increasing sequence; hence it has to leave the set $|z_t| < \delta$. This proves our assertion. Summarizing we have the following:

1) If $0 \le a < 1$, the equilibrium at $\hat{x}^1 = 0$ is attractive and the equilibrium at $\hat{x}^2 = (a-1)/a$ is repulsive.
2) If $1 < a < 3$, the equilibrium at $\hat{x}^1 = 0$ is repulsive while the equilibrium at $\hat{x}^2 = (a-1)/a$ is attractive.
3) If $a > 3$ both equilibria are repulsive.

B. Other types of solutions are the so-called cyclic solutions satisfying $x_{t+T} \equiv x_t$ for some positive integer T. Let $T = 2$; i.e., $x_{t+2} = x_t$. In the case of (1.1) we shall have

$$x_{t+2} = ax_{t+1}(1 - x_{t+1}) = a^2 x_t (1 - x_t)[1 - ax_t(1 - x_t)] \equiv x_t$$

The initial conditions defining cycles are the solutions of

$$x = a^2 x (1-x)[1 - ax(1-x)] \tag{1.8}$$

that is the fixed points of the mapping

$$f^2(x) = (f \circ f)(x) = a^2 x (1-x)[1 - ax(1-x)]$$

the iterated mapping. Obviously the fixed points of $f(x)$ are among the fixed points of $f^2(x)$; they define equilibria not cycles so we ignore them. It is easily seen that the solutions of interest of (1.8) are the solutions of the equation

$$a^2 x^2 - a(a+1)x + a + 1 = 0,$$

namely,

$$\hat{x}^3 = \frac{a + 1 - \sqrt{(a+1)(a-3)}}{2a}, \quad \hat{x}^4 = \frac{a + 1 + \sqrt{(a+1)(a-3)}}{2a}.$$

These solutions define the 2-cycle in the sense that

$$x_t = \hat{x}^3, \; x_{t+1} = \hat{x}^4, \; x_{t+2} = \hat{x}^3.$$

One may ask whether this 2-cycle is attractive, i.e., whether x_t with x_0 in a neighborhood of $\hat{x}^3$ or $\hat{x}^4$ will approach $\hat{x}^3, \hat{x}^4$, respectively.

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