国外数学名著系列（续一）

（影印版）59

R. Osserman (Ed.)

# Geometry V
## Minimal Surfaces

# 几 何 V
## 最小曲面

科学出版社
www.sciencep.com

国外数学名著系列(影印版) 59

# Geometry V

## Minimal Surfaces

# 几 何 V

## 最小曲面

R. Osserman (Ed.)

科 学 出 版 社

北 京

图字：00-2008-5377

R. Osserman(Ed.): Geometry V: Minimal Surfaces


**图书在版编目(CIP)数据**

几何V：最小曲面 = Geometry V: Minimal Surfaces / (德)奥斯曼 (Osserman, R.) 编著. —影印版. —北京：科学出版社，2009

(国外数学名著系列；59)

ISBN 978-7-03-023502-2

I. 几… II. 奥… III. ①几何-英文 ②极小曲面-英文 IV. O18

中国版本图书馆 CIP 数据核字(2008) 第 186176 号

责任编辑：范庆奎/责任印刷：钱玉芬/封面设计：黄华斌

科学出版社 出版
北京东黄城根北街16号
邮政编码：100717
http://www.sciencep.com
双青印刷厂 印刷
科学出版社发行 各地新华书店经销
*
2009年1月第 一 版 开本：B5(720×1000)
2009年1月第一次印刷 印张：18
印数：1—2 500 字数：343 000

**定价：66.00 元**

(如有印装质量问题，我社负责调换〈双青〉)

# 《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

# List of Editors and Authors

*Editor-in-Chief*

R.V. Gamkrelidze, Russian Academy of Sciences, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow; Institute for Scientific Information (VINITI), ul. Usievicha 20a, 125219 Moscow, Russia; e-mail: gam@ipsun.ras.ru

*Consulting Editor*

R. Osserman, Deputy Director, Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, USA; e-mail: osserman@msri.org

*Authors*

H. Fujimoto, Department of Mathematics, Faculty of Sciences, Kanazawa University, Kakuma-machi, 920-11 Kanazawa, Japan; e-mail: fujimoto@kappa.s.kanazawa-u.ac.jp

S. Hildebrandt, Mathematisches Institut, Rheinische Friedrich-Wilhelms Universität Bonn, Beringstr.1, D-53115 Bonn, Germany

D. Hoffman, Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, USA; e-mail: david@msri.org

H. Karcher, Mathematisches Institut, Rheinische Friedrich-Wilhelms Universität Bonn, Beringstr.1, D-53115 Bonn, Germany; e-mail: unm416@ibm.rhrz.uni-bonn.de

L. Simon, Department of Mathematics, Stanford University, Stanford, CA 94305, USA; e-mail: lms@math.stanford.edu

This volume is dedicated to

Lars V. Ahlfors

1907–1996

twentieth century giant of geometric function theory,
whose influence is clearly visible here, as in many other parts
of contemporary mathematics.

# Contents

# Introduction

Few people outside of mathematics are aware of the varieties of mathematical experience – the degree to which different mathematical subjects have different and distinctive flavors, often attractive to some mathematicians and repellant to others.

The particular flavor of the subject of minimal surfaces seems to lie in a combination of the concreteness of the objects being studied, their origin and relation to the physical world, and the way they lie at the intersection of so many different parts of mathematics. In the past fifteen years a new component has been added: the availability of computer graphics to provide illustrations that are both mathematically instructive and esthetically pleasing.

During the course of the twentieth century, two major thrusts have played a seminal role in the evolution of minimal surface theory. The first is the work on the Plateau Problem, whose initial phase culminated in the solution for which Jesse Douglas was awarded one of the first two Fields Medals in 1936. (The other Fields Medal that year went to Lars V. Ahlfors for his contributions to complex analysis, including his important new insights in Nevanlinna Theory.) The second was the innovative approach to partial differential equations by Serge Bernstein, which led to the celebrated Bernstein's Theorem, stating that the only solution to the minimal surface equation over the whole plane is the trivial solution: a linear function.

The subsequent history of these two problems provides a fascinating glimpse into the way a single result can provide the seed for a profusion of growth in many directions. In the case of Bernstein's Theorem, the two obvious directions for generalization were toward a wider class of equations, and to higher dimensions. Both of those took a while to be realized, since Bernstein's original proof did not lend itself easily to generalizations of any sort. The first step was the discovery of alternative proofs, particularly by Bers, Heinz, and Nitsche. They made possible the introduction of equations of "minimal surface type" with many of the properties of the minimal surface equation, including the Bernstein Theorem. These results are described below in the article by Leon Simon, himself one of the leading contributors to those developments. Heinz's proof involved a generalization in a different direction: bounds on the second derivatives at a point for any solution defined over a disk centered at the point. Those bounds tend to zero as the radius of the disk tends to infinity, showing that a solution over the whole plane must have all second derivatives identically zero and therefore be a linear function. The second-derivative bounds can also be reformulated as a bound on the Gauss curvature, and that approach proved useful in extensions to higher dimensions. Yet another approach to Bernstein's Theorem, due to Fleming, provided the key ideas that yielded the first results for higher dimensions. The big surprise was the discovery that Bernstein's Theorem turned out to be true for solutions in up to seven variables, and then false. That again led to a whole series of new questions about global behavior of the minimal surface equation in higher dimensions, all of which are explored in the article by Simon.

A generalization of Bernstein's Theorem in quite another direction was suggested by Nirenberg. His idea was to think of the theorem in more geometric terms, concerning surfaces not so much as solutions of a differential equation, but as surfaces with zero mean curvature, and replacing the assumption that the surface projects onto the whole plane by the pair of assumptions that the surface be complete and the Gauss map omit a neighborhood of some point. Formulated in those terms, the result was reminiscent of the Weierstrass Theorem stating that the values of a nonconstant entire function must be everywhere dense. That led Nirenberg to a second conjecture that there should also be a Picard Theorem for minimal surfaces, saying that for a complete minimal surface, if the Gauss map omits more than two values, then the Gauss map must be constant, and the surface must be a plane. It turned out that his first conjecture was correct, and the second one false. But it was Nirenberg's formulation that was most crucial to further developments, since it led to a whole new theory, that of complete minimal surfaces, first in three dimensions, and then of arbitrary codimension. Also, it was natural to think of replacing Weierstrass and Picard-type results by a Nevanlinna Theory for the Gauss map. An early result in that direction was the theorem of Ahlfors-Osserman that the Gauss map of a complete nonplanar minimal surface not only could not be bounded, but could not belong to Nevanlinna's class of "bounded type". That provided a considerable strengthening of the Weierstrass result that the Gauss map must be everywhere dense to the fact that the set of omitted points had to have logarithmic capacity zero. But the full flowering of the Nevanlinna theory for complete minimal surfaces came with the work of Fujimoto, and is explained in detail in his paper in this volume. Fujimoto was also the one to provide the definitive answer to the second Nirenberg conjecture. After Xavier showed that the image under the Gauss map could omit at most a finite number of points, Fujimoto proved that the precise maximum for the number of omitted points possible was four.

The theory of complete minimal surfaces blossomed in quite another direction following the work of Jorge and Meeks devoted to analyzing the possibilities for a complete minimal surface to be embedded in 3-space, with no self intersections. Before 1980, the only known examples of complete embedded minimal surfaces were the plane, the catenoid, and various periodic surfaces, such as the helicoid. The latter necessarily had infinite total curvature, and it seemed not unlikely that there were no others of finite total curvature. Then Costa came up with an example of a surface which was of genus one, with three embedded ends, and opened the floodgate for further examples, after Hoffman and Meeks showed that Costa's surface was in fact embedded. In the decade since the publication of Costa's example and Hoffman and Meeks' first paper, the subject has seen phenomenal growth, all of which is described in the paper by Hoffman and Karcher in this volume.

These are by no means the only developments that can be traced directly back to Bernstein's seminal theorem, but they illustrate the manner in which a single result can have a powerful influence on the future course of mathematics.

The Plateau problem has been at least equally influential. The general existence theorems of Douglas left some important questions unanswered. First, Douglas was unable to rule out the possibility that the surfaces obtained had certain kinds of singularities, called branch points. Second, he left open the question of regularity of the surface at the boundary. And third, there was the question of how many different solution surfaces might be bounded by a single curve. Much progress has been made on these questions, but much also remains to be done. A full account of the results to date is given in the paper by Hildebrandt in this volume.

One of the most important roles of Plateau's problem was as a spark to the development of the powerful new tool known as geometric measure theory. There were two rather different approaches, one due basically to de Giorgi, and the other to Federer and Fleming. After leading separate lives for a while, both from each other and from the "classical" approach to minimal surfaces, all three methods gradually intermingled, so that methods and results from geometric measure theory find their way into a number of topics discussed in this book. Perhaps surprisingly, it has proved especially critical in the PDE aspect of the subject, where it has played a major role in connection with Bernstein's Theorem.

Taken together, the articles in this volume provide a fairly broad spectrum of recent activity in the field of minimal surfaces. Needless to say, it is far from comprehensive. The lists of references at the end of the different articles point in a number of further directions. A combined index to all four articles at the end of this volume will allow a reader to track down topics of particular interest, sometimes seen from different perspectives in different articles. That should help to provide at least some of the flavor that has given the subject of minimal surfaces such an enduring appeal over the years.

January 1997 *Robert Osserman*

# I. Complete Embedded Minimal Surfaces of Finite Total Curvature

David Hoffman[1] and Hermann Karcher[2]

## Contents



---

[1] Supported by research grant DE-FG03-95ER25250 of the Applied Mathematical Science subprogram of the Office of Energy Research, U.S. Department of Energy, and National Science Foundation, Division of Mathematical Sciences research grant DMS-9596201. Research at MSRI is supported in part by NSF grant DMS-90-22140.

[2] Partially supported by Sonderforschungsbereich SFB256 at Bonn.

# 1. Introduction

We will survey what is known about minimal surfaces $S \subset \mathbf{R}^3$, which are complete, embedded, and have finite total curvature: $\int_S |K| dA < \infty$. The only classically known examples of such surfaces were the plane and the catenoid. The discovery by Costa [14, 15] early in the last decade, of a new example that proved to be embedded sparked a great deal of research in this area. Many new examples have been found, even families of them, as will be described below. The central question has been transformed from whether or not there are any examples except surfaces of rotation to one of understanding the structure of the space of examples.

Up to this point, *every new example* of a complete, embedded minimal surface of finite total curvature has been discovered first by using the global version of the Enneper-Riemann-Weierstrass representation, which is essentially due to Osserman [58, 59].[3] This involves knowledge of the compact Riemann surface structure of the minimal surface, as well as its Gauss map and other geometric-analytic data. One of our goals is to show how this is done in the case that has been most completely analyzed, namely surfaces with genus $\geq 1$ and three topological ends. An important quality of this construction is that the Riemann surface and the meromorphic data are constructed simultaneously under the assumption of symmetry. Moreover, once this is done, parameters must be found in order to have a well-defined finite-total-curvature surface. This parameter search is typically done by computer using a combination of relatively simple numerical routines and relatively complex graphics tools ([8, 41]). In many cases a full theoretical analysis, as is done here in Section 4 for the three-ended surfaces of Theorem 3.3 has yet to be carried out. Moreover, solution of the period problem does not at all guarantee that the surfaces are embedded. In Section 5 we present examples of Callahan, Hoffman, Karcher, Meeks and Wohlgemuth that lie in one parameter families containing both embedded and immersed surfaces. In fact the period problem and embeddedness are totally independent issues. There are examples of Weierstrass data meeting all necessary conditions (Proposition 2.4) for embeddedness, for which the period problem is not solvable (a genus-one example with two catenoid ends does not exist but Weierstrass data for such a surface – even a very symmetric one – does) and others for which the period

[3] *Added in Proof* (November, 1996). This was written in early 1995. In the Fall of 1995, Nikolaos Kapouleas constructed new examples using methods similar to those he used to construct higher genus surfaces of constant mean curvature. Briefly, he proves that the construction imagined and described at the beginning of Section 5.2 can be carried out in detail (Kapouleas, N.: Complete embedded minimal surfaces of finite total curvature, preprint). Martin Traizet has carried out an analogous construction for periodic minimal surfaces (Traizet, M.: Construction de surfaces minimales en recollant des surfaces de Scherk, Annales de l'Institut Fourier (to appear)).

problem is solvable for a family of surfaces that are embedded outside of a compact set of $\mathbf{R}^3$, but are not embedded.

The survey is organized as follows. In Section 2 we present the basic tools of the subject, the most important of which is the Weierstrass-Enneper representation. In Section 2.2 we describe the construction of Chen and Gackstatter [10], which produces a genus-one surface with the symmetries and end behavior of Enneper's surface. To our knowledge this complete minimal surface of finite total curvature was the first one explicitly constructed by first specifying a geometric property – in this case, end behavior – and then deriving the necessary Weierstrass data. It is not, of course, embedded but its construction, as presented here, has most of the features of the construction of the three-ended examples in Section 4. In Section 2.3, the hypothesis of embeddedness is used to derive relationships between the geometry of the surface and its analytic representation. Propositions 2.3, 2.4 and 2.5 gather together all necessary conditions including the relationship between flux, logarithmic growth rates of the ends and residues of the complex differential of the height function.

In Section 3, we present the few global rigidity theorems that are known. (Theorems 3.1 and 3.4, due to Lopez-Ros [51], Schoen [67] and Costa [16].) We present a proof (in Section 3.1) of the Lopez-Ros theorem, which states that a complete minimal surface of genus zero and finite total curvature must be the plane or the catenoid. Our proof follows that of Perez-Ros [60]. We also state the existence result, Theorem 3.3, for the three-ended, complete minimal surfaces with genus $k-1$ and $k$ vertical planes of symmetry ([30]). The details of the construction of these surfaces are presented in Section 4. We include here the estimation of the parameters that solve the period problem when $k > 2$. The values of the parameters that close the periods determine the logarithmic growth rates of the ends of these surfaces. For the surface to be embedded, they must lie in a certain range, which they (happily) do. This is done in Sections 4.5–4.9.

In Section 5, we survey other known examples and discuss what little is known about the structure of the space of complete embedded minimal surfaces of finite total curvature. Section 5.2 presents some conjectures about this space.

Finite total curvature implies finite topology, even without the additional assumption of embeddedness. The converse is not true; the helicoid is simply connected, nonflat and periodic, so its curvature is infinite, while its topology is finite. Up until recently, the helicoid was the only known embedded minimal surface with finite topology and infinite total curvature. In 1992, we discovered, with Fusheng Wei, a complete embedded minimal surface of genus one with one end – asymptotic to the helicoid – that has infinite total curvature [32, 33]. The details of this construction are outside the scope of this work.

However, the extent to which finite topology implies finite total curvature is discussed in Section 6.[4]

Section 7 discusses the index of stability of a complete minimal surface. The basic results [22, 25, 26] the equivalence of finite index and finite total curvature, and the fact that the index is completely defined by the Gauss map, are discussed. There is, as yet, no known relationship between embeddedness and properties of the index. This final section is therefore, strictly speaking, misplaced in this survey. However the ideas and techniques may, in the long run, prove useful in the study of embeddedness of minimal surfaces.

*Acknowledgements.* The authors would like to thank Jim Hoffman of MSRI and Ed Thayer, now in the Department of Molecular Biotechnology at the University of Washington, for the preparation of the computer graphics illustrations in this article. Both were formerly at the Center for Geometry, Analysis, Numerics and Graphics (GANG) at the University of Massachusetts, Amherst where one of us was the Director. Peter Norman and Pascal Romon read earlier versions and gave us useful critical comments. Harold Rosenberg helped clarify some important points in Section 2.3. We are grateful to M. Kotani and S. Nayatani for useful comments on Section 7. Robert Bryant and Michael Wolf gave us many useful comments. The authors enjoyed the hospitality of MSRI where most of the final work on this paper was done.

## 2. Basic Theory and the Global Weierstrass Representation

Let $\mathcal{S}, \langle\rangle$ be an oriented surface with a Riemannian metric, and let $\nabla$ denote its Riemannian connection. For a smooth function $f\colon \mathcal{S} \to \mathbf{R}$, we will denote its differential by $df$ and its gradient by grad $f$. They are related by $Vf = df(V) = \langle \operatorname{grad} f, V\rangle, V \in T\mathcal{S}$, which can be thought of as the defining equation for grad $f$. If $w$ is a one form, its covariant derivative $\nabla w$ is defined by the relation

$$(\nabla_U w)V = Uw(V) - w(\nabla_U V)\,. \tag{2.1}$$

The divergence operator is $\operatorname{div} = \operatorname{tr} \nabla$ and the Laplacian of a smooth function $f$ is given by $\Delta f := \operatorname{div} \operatorname{grad} f$. Note that

$$\langle \nabla_U \operatorname{grad} f, V\rangle = U\langle \operatorname{grad} f, V\rangle - (\nabla_U V)f = (\nabla_U df)(V)\,. \tag{2.2}$$

[4] *Added in Proof* (November, 1996). Pascal Collin recently showed that a complete embedded minimal surface with finite topology and *more than one end* has to have finite total curvature. He did this by proving the Nitsche Conjecture. See Section 6.2. (Collin, P.: Topologie et courbure des surfaces minimales proprement plongées de $R^3$, Ann. Math. **145** (1997), 1–31.