LECTURES ON FOURIER INTEGRALS

BY

Salomon Bochner

WITH AN AUTHOR'S SUPPLEMENT ON

Monotonic Functions,

Stieltjes Integrals,

and Harmonic Analysis

TRANSLATED FROM THE ORIGINAL BY

Morris Tenenbaum

and

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TRANSLATORS' PREFACE

In undertaking this translation of Bochner's classical book and its supplement (Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse, Mathematische Annalen, Volume 108 (1933), pp. 378-410), our main purpose was to make generally available to the present generation of group-theorists and practitioners in distributions the historical and concrete problems which gave rise to these disciplines. Here can be found the theory of positive definite functions, of the generalized Fourier integral, and even forms of the important theorems concerning the reciprocal of Fourier transforms.

The translators are grateful to Professor Bochner for his encouragement in this work and for his many valuable suggestions.

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CHAPTER I

BASIC PROPERTIES OF TRIGONOMETRIC INTEGRALS

§1. Trigonometric Integrals Over Finite Intervals

1. We denote as trigonometric integrals expressions of the form

$$\Phi(\alpha) = \int_a^b f(x) \cos \alpha x \, dx \tag{1}$$

or

$$\Psi(\alpha) = \int_a^b f(x) \sin \alpha x \, dx \quad . \tag{2}$$

It is frequently more convenient to use the exponential factor $e^{i\alpha x}$ in place of the trigonometric factors $\cos \alpha x$ and $\sin \alpha x$. The trigonometric integral will then read

$$J(\alpha) = \int_a^b f(x) \, e^{i\alpha x} \, dx$$ [1] .

For typographical simplification we shall always denote the function $e^{i\xi}$ by $e(\xi)$. Hence we shall write $J(\alpha)$ as

$$J(\alpha) = \int_a^b f(x) e(\alpha x) \, dx \quad . \tag{3}$$

It is also customary to denote trigonometric integrals as <u>Fourier integrals</u> [1] because J. J. Fourier provided the first incentive to the study of these integrals [2].

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[1] We shall also frequently use the symbols Γ, H, J to denote respectively the gamma function, the Hankel function and the Bessel function. These special uses at times will be evident from the context.

Whenever the contrary is not evident from the context, a "number" will be a complex number, and a function a complex function of a real variable. A function $f(x)$ will therefore be an expression of the form $f_1(x) + if_2(x)$ where $f_1(x)$ and $f_2(x)$ are real valued functions as usually defined. For dealing with such functions, cf. Appendix 12. We shall assume once and for all that each function which occurs under an integral sign, will first of all be integrable on each finite interval, and we shall take as a basis the integral concept of Lebesque. Thus we assume that, automatically, any given function is measurable (Lebesgue) in its entire extent and "summable" (Lebesgue) in every finite interval.

2. If the limits of integration a and b are the same for the integrals (1) to (3), then

$$J(\alpha) = \Phi(\alpha) + i\Psi(\alpha) \quad ,$$

and

$$2\Phi(\alpha) = J(\alpha) + J(-\alpha); \qquad 2i\Psi(\alpha) = J(\alpha) - J(-\alpha) \quad .$$

Because of the similarity in construction of $\Phi(\alpha)$ and $\Psi(\alpha)$, we shall frequently prove a statement for only one of the three integrals, and when the transfer is an obvious one, assume its correctness for the other two. Since, in addition,

$$\Phi(-\alpha) = \Phi(\alpha); \qquad \Psi(-\alpha) = -\Psi(\alpha)$$

and

$$J(-\alpha) = \overline{J_1(\alpha)} \quad ,$$ [1]

where

$$J_1(\alpha) = \int_a^b \overline{f(x)} e(\alpha x)\, dx$$

with the function $f_1(x) = \overline{f(x)}$, it will suffice for the study of the functions $\Phi(\alpha)$, $\Psi(\alpha)$ and $J(\alpha)$ to limit ourselves to one of the half lines $\alpha \geq 0$ or $\alpha \leq 0$. As a rule, we shall favor the right half line.

3. At least one of the two limits of the definite integral with which we shall be concerned, will in general, be infinite. To simplify writing, we shall omit the upper integration limit when its value is $+\infty$, and the lower limit when its value is $-\infty$. The integral

$$\int_0 f(x) \cos \alpha x\, dx$$

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[1] The crossbar means "conjugate-complex".

will therefore extend, for example, over the interval $[0, \infty]'$, and the integral

$$\int f(x)e(\alpha x)\,dx \tag{4}$$

over the interval $[-\infty, \infty]$.

4. A basic property of the trigonometric integrals with which we shall immediately concern ourselves is that they become, "in general", arbitrarily small for large values of α. In this section we shall restrict ourselves to the case where the limits of integration are finite, and only to a discussion of the integral (3).

If the function $f(x)$ has no qualifications attached to it, then the integral (3) is merely a special case of (4). The integral (3) becomes (4) if, outside of (a, b), the function $f(x)$ is extended by means of zero values; i.e., if $f(x)$ is extended, by means of the stipulation "$f(x) = 0$ if $x \neq (a, b)$", to a function defined in $[-\infty, \infty]$. This assertion is not true, however, if $f(x)$ has restrictions assigned to it. For example, if it is required that $f(x)$ be differentiable, then (3) is a special case of (4) only if $f(x)$ vanishes for $x = a$ and $x = b$; since only then is the new function which arises by assigning zero values outside (a, b) differentiable in $[-\infty, \infty]$ (cf. Appendix 8). And if $f(x)$, after its extension in $[-\infty, \infty]$ is intended to be continuously differentiable, then not only must it be continuously differentiable in (a, b) but the function and its derivative must also both vanish for $x = a$ and $x = b$.

Our assertion states that[2]

$$J(\alpha) \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \pm\infty \,. \tag{5}$$

If $f(x)$ is differentiable in (a, b) and if we denote by M, a bound of $f(x)$ and also of

$$\int_a^b |f'(x)|dx \,,$$

then it follows from

[1] (λ, μ) will mean the interval $\lambda \leq x \leq \mu$; $[\lambda, \mu]$ the interval $\lambda < x < \mu$. Mixed brackets will also be employed so that $(\lambda, \mu]$ will mean $\lambda \leq x < \mu$.

[2] We shall write for the limit, with no difference in meaning, either $\lim f(\xi) = h$ or $f(\xi) \longrightarrow h$.

$$J(\alpha) = \frac{1}{i\alpha}\,[f(b)e(\alpha b) - f(a)e(\alpha a)] - \frac{1}{i\alpha}\int_a^b f'(x)e(\alpha x)\,dx$$

that

$$(6) \qquad |J(\alpha)| \leqq \frac{3M}{|\alpha|}$$

and from this (5) follows. If we write

$$J(\alpha) = \int_a^c + \int_c^b = J_1(\alpha) + J_2(\alpha)$$

and if (5) is valid for $J_1(\alpha)$ and $J_2(\alpha)$, then it is evidently also valid for $J(\alpha)$. A similar reasoning would apply if more intervals were involved. Hence (5) is valid for a piecewise differentiable function, in particular for a piecewise constant function ("step function").

By a limiting process it is now possible to prove that (5) is valid for any (integrable) function. Let $f(x)$ and $f_1(x)$ be two functions such that

$$(7) \qquad \int_a^b |f(x) - f_1(x)|dx \leqq \epsilon \quad .$$

Then for the corresponding integrals $J(\alpha)$ and $J_1(\alpha)$, one has

$$|J(\alpha) - J_1(\alpha)| = \left|\int_a^b (f(x) - f_1(x))e(\alpha x)\,dx\right|$$

$$\leqq \int_a^b |f(x) - f_1(x)|dx \leqq \epsilon \quad .$$

Let (5) be satisfied for $f_1(x)$. Hence there exists an $\alpha(\epsilon)$ such that for $|\alpha| > \alpha(\epsilon)$

$$|J_1(\alpha)| \leqq \epsilon \quad .$$

Therefore for $|\alpha| \geqq \alpha(\epsilon)$,

$$|J(\alpha)| \leqq |J_1(\alpha)| + |J(\alpha) - J_1(\alpha)| \leqq 2\epsilon \quad .$$

But to each (integrable) function $f(x)$ and to each ϵ, one can specify a step-function $f_1(x)$ which satisfies (7), (Appendix 10). From this it follows that:

For each function[1] $f(x)$

$$\int_a^b f(x)e(\alpha x)\,dx \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \pm\infty .$$

An analogous relation also holds for the functions $\Phi(\alpha)$ and $\Psi(\alpha)$ [3].

5. We observe that $J(\alpha)$ is a continuous function, and this fact can be proved as follows:

$$|J(\alpha + \rho) - J(\alpha)| \leq \int_a^b |f(x)||e(\rho x) - 1|\,dx \leq M(\rho) \int_a^b |f(x)|\,dx ,$$

where $M(\rho)$ is the maximum of $|e(\rho x) - 1|$ in the interval (a, b). But if $\rho \longrightarrow 0$, then $M(\rho) \longrightarrow 0$. — $\Phi(\alpha)$ and $\Psi(\alpha)$ are also continuous functions, cf. 2[2].

§2. Trigonometric Integrals Over Infinite Intervals.

1. We say that the function $g(x)$ is integrable in $[a, \infty]$, if the integral

$$\int_a^A g(x)\,dx$$

approaches a finite limit as $A \longrightarrow \infty$. We denote this limit by

(1) $$\int_a g(x)\,dx .$$

We shall also say that the integral (1) "exists" or that it "converges".

---

1 Since the function $f(x)$ occurs under the integral sign, it will be tacitly assumed, as agreed to in our previous statement, that it is integrable.

2 Paragraph 2 of the present section is meant. Each section is divided into several paragraphs. A simple number denotes a paragraph, and a round bracketed number a formula. Therefore (5) denotes the formula (5). If the paragraph or formula is quoted from other sections, then the number of the section is stated in advance. Thus §51, 3 denotes paragraph 3 of §51, and §51, (9), the formula (9) of §51.

Whenever a function $g(x)$ has a certain property in a subinterval $[A, \infty]$ or $[-\infty, B]$ of its interval of definition, then we shall also say that it has this property as $x \longrightarrow \infty$ or as $x \longrightarrow -\infty$.

Since for each $A > a$, the integral (1) along with

(2) $$\int_A g(x)\, dx$$

either converges or does not converge, it follows that the function $g(x)$ is integrable in $[a, \infty]$ if it is integrable as $x \longrightarrow \infty$.

It is a basic property of the Lebesgue integral, that in a finite interval, each integrable function is also absolutely integrable. Hence each of the functions considered heretofore is, in each finite interval, absolutely integrable. The same assertion, however, cannot be made if the interval of integration is infinite. If $g(x)$ is integrable as $x \longrightarrow \infty$ in the sense of our definition, then $|g(x)|$ need not be also integrable as $x \longrightarrow \infty$, although the converse does hold. Next, if $f(x)$ is absolutely integrable in $[a, \infty]$ then because $|f(x) \sin \alpha x| \leqq |f(x)|$, the integral

(3) $$\Psi(\alpha) = \int_a f(x) \sin \alpha x\, dx$$

converges for all values of α. Again $\Psi(\alpha) \longrightarrow 0$ as $\alpha \longrightarrow \pm\infty$. This is deducible from

$$|\Psi(\alpha)| \leqq \left| \int_a^A f(x) \sin \alpha x\, dx \right| + \int_A |f(x)|\, dx .$$

Since the second integral on the right is independent of α, it can be made, by a suitable choice of A, smaller than ϵ. With A fixed, the first integral will become smaller than ϵ for $|\alpha| \geqq \alpha(\epsilon)$. Hence for $|\alpha| \geqq \alpha(\epsilon)$

$$|\Psi(\alpha)| \leqq 2\epsilon \quad .$$

Corresponding assertions are valid for $\Phi(\alpha)$ and $J(\alpha)$.

For example, let $f(x) = e^{-kx}$, $k > 0$ and $a = 0$, and let us calculate $J(\alpha)$, the simplest of the three. From

$$\int_0^A e^{-(k-i\alpha)x}\, dx = \frac{1}{k - i\alpha}\left(1 - e^{-kA} e(\alpha A)\right) \quad ,$$

one obtains, by letting $A \longrightarrow \infty$, and by separating the real and imaginary parts

$$(4) \qquad \int_0 e^{-kx} \cos \alpha x \, dx = \frac{k}{k^2 + \alpha^2}; \qquad \int_0 e^{-kx} \sin \alpha x \, dx = \frac{\alpha}{k^2 + \alpha^2} .$$

Both expressions actually approach zero as $\alpha \longrightarrow \pm \infty$ [4].

As regards behavior at infinity, an important class of functions which need not be absolutely integrable are monotonic functions. Let the (real valued) function $f(x)$ converge monotonically to zero as $x \longrightarrow \infty$, i.e., let it be monotonic in a certain interval $[A, \infty]$, and convergent to zero as $x \longrightarrow \infty$. Since we already have at our command integrals over finite intervals, we can assume, therefore, that the point $x = A$ coincides with the initial point $x = a$. A function monotonic in $[a, \infty]$ which converges to zero as $x \longrightarrow \infty$ is, in its entire range, either positive and decreasing, or negative and increasing. Since an increasing function becomes, by a change of sign, a decreasing one, we need consider only the decreasing one.

2. We shall need the following theorem of analysis; the so-called second mean value theorem of integration. If, in the interval (a, b), the function $\varphi(x)$ is continuous, and the function $p(x)$ is positive and monotonically decreasing, then in the interval (a, b) there is a value c between a and b for which

$$\int_a^b p(x)\varphi(x) \, dx = p(a) \int_a^c \varphi(x) \, dx .$$

In particular, let $\varphi(x) = \sin \alpha x$, $\alpha > 0$. From

$$\left| \int_a^c \sin \alpha x \, dx \right| \leq \frac{2}{\alpha}$$

it follows that

$$(5) \qquad \left| \int_a^b p(x) \sin \alpha x \, dx \right| \leq \frac{2p(a)}{\alpha} .$$

Now in $(a, \infty]$, let the function $p(x)$ decrease monotonically to zero. From

$$(6) \qquad \left| \int_A^{A'} p(x) \sin \alpha x \, dx \right| \leq \frac{2}{\alpha} p(A) \qquad \alpha > 0,$$

in conjunction with the fact that $p(A) \longrightarrow 0$ as $A \longrightarrow \infty$, it follows that the integral

$$\Psi(\alpha) = \int_a p(x) \sin \alpha x \, dx$$

is convergent for $\alpha > 0$. We can now allow A' in (6) to become infinite, and we have

$$\left| \int_A p(x) \sin \alpha x \, dx \right| \leqq \frac{2}{\alpha} p(A) \quad .$$

Hence it follows that $\Psi(\alpha) \longrightarrow 0$ as $\alpha \longrightarrow \infty$. Summarizing, we formulate the following theorem.

THEOREM 1. If in $[a, \infty]$, the function $f(x)$ under consideration, as $x \longrightarrow \infty$; either

1. is absolutely integrable, or
2. converges monotonically to zero,

then the integrals $\Phi(\alpha)$, $\Psi(\alpha)$, $J(\alpha)$ exist for

1. all α or
2. all $\alpha \neq 0$,

and converge to zero as $\alpha \longrightarrow \pm \infty$ [5].

The restriction $\alpha \neq 0$, made under 2 applies only to $\Phi(\alpha)$ and $J(\alpha)$. For a function decreasing monotonically to zero, it is not necessary that the integral

$$\int_a f(x) \, dx \ ,$$

which should represent the value $\Phi(0)$ or $J(0)$, converge (for example $f(x) = 1/x$).

Now let $f(x)$ be representable in the form

$$f(x) = g(x) \sin px \quad ,$$

where p is a constant, and $g(x)$ approaches zero monotonically. By means of the relation

$$2\Psi(\alpha) = \int_a g(x) \cos (\alpha - p)x \, dx - \int_a g(x) \cos (\alpha + p)x \, dx \quad ,$$

one recognizes again that, with the possible exceptions of $\alpha = p$ and

$\alpha = -p$, the integral $\Psi(\alpha)$ exists and converges to zero as $\alpha \longrightarrow \pm \infty$. The same assertion holds for

$$f(x) = g(x) \cos px \quad ,$$

and more generally for

$$f(x) = g(x) \sin (px + q) \quad ,$$

where p and q are constants.

THEOREM 1a. The assumption 2 in Theorem 1 can be generalized by setting

$$f(x) = g(x) \sin (px + q) \quad ,$$

where p and q are constants, and $g(x)$ approaches zero monotonically as $x \longrightarrow \infty$. However, the integrals need not converge for the values $\alpha = \pm p$ [5].

THEOREM 1b. A further generalization of the theorem results if the factor $\cos \alpha x$ or $\sin \alpha x$ in $\Phi(\alpha)$ and $\Psi(\alpha)$, is replaced by $\cos \alpha(x - t)$ or $\sin \alpha(x - t)$, where t is an additional constant [5].

This generalization can be justified by the transformation $y = x - t$.

3. Analogous statements are valid for a left half line $[-\infty, b]$, and for the entire interval $[-\infty, \infty]$. We call the integral

$$\int g(x)\, dx$$

convergent if both integrals

$$\int_0 g(x)\, dx \quad \text{and} \quad \int^0 g(x)\, dx$$

converge. In this sense, we shall later on attach to the function $f(x)$, the special integral

$$E(\alpha) = \frac{1}{2\pi} \int f(x) e(-\alpha x)\, dx$$

and denote it as the (<u>Fourier</u>) <u>transform</u> [6] of $f(x)$. The integral $E(\alpha)$

is therefore normalized somewhat differently from the integral $J(\alpha)$, namely

$$J(\alpha) = 2\pi\, E(-\alpha) \quad .$$

From the above, we see that $E(\alpha)$ exists for all $\alpha \neq 0$, and converges to zero as $\alpha \longrightarrow \pm\infty$, provided $f(x)$ is either absolutely convergent or approaches zero monotonically not only as $x \longrightarrow \infty$ but also as $x \longrightarrow -\infty$.

4. If $a > 0$, then the integral

$$\int_a^\infty \frac{\sin \alpha x}{x}\, dx$$

falls under Theorem 1, because in the interval $(a, \infty]$, the function $f(x) = 1/x$ decreases monotonically to zero. On the other hand $f(x)$ is not integrable in the interval $[0, a]$, and therefore not in $[0, \infty]$. Although the whole integrand $x^{-1}\sin \alpha x$ is regular there, and hence the integral

$$\Psi(\alpha) = \int_0^\infty \frac{\sin \alpha x}{x}\, dx$$

exists for all α, yet $\Psi(\alpha)$ is not convergent to zero as $\alpha \longrightarrow \infty$. The transformation $\alpha x = \xi$, $\alpha > 0$ yields for example

$$\int_0^\infty \frac{\sin \alpha x}{x}\, dx = \int_0^\infty \frac{\sin \xi}{\xi}\, d\xi \quad .$$

Hence $\Psi(\alpha)$ is constant for $\alpha > 0$, and this constant, as we shall see later in §4, 3 is different from zero.

§3. Order of Magnitude of Trigonometric Integrals

1. The question arises whether an assertion can be made with regard to the rapidity with which $\Phi(\alpha)$ and $\Psi(\alpha)$ decrease to zero as $\alpha \longrightarrow \infty$. According to Lebesgue, if the function $f(x)$ is only known to be (absolutely) integrable, no statement of this kind can be made even if the interval happens to be finite. Rather, it can be shown that these integrals can decrease to zero arbitrarily slowly [7]. The situation changes however, if more precise information about the function $f(x)$ is available. If $f(x)$ is monotonically decreasing in (a, b) or monotonically decreasing to zero in $(a, \infty]$, then by §2, (5), there exists a constant A, such that for $\alpha > 0$

$$|\Psi(\alpha)| \leq A\,\alpha^{-1}$$

which can be written with the familiar Landau symbol

(1) $$\Psi(\alpha) = O(\alpha^{-1}) \ .$$

We recall the meaning of this symbol. Let $\varphi(\xi) > 0$ as $\xi \longrightarrow \infty$. Then $f(\xi) = O[\varphi(\xi)]$ states that the quotient

$$f(\xi)/\varphi(\xi)$$

is bounded as $\xi \longrightarrow \infty$; and $f(\xi) = o[\varphi(\xi)]$ states that it approaches zero. If $f(\xi) = O[\varphi(\xi)]$, and $f_1(\xi) = O[\varphi_1(\xi)]$, where $\varphi(\xi) \leq \varphi_1(\xi)$, and if $h(\xi)$ and $h_1(\xi)$ are bounded as $\xi \longrightarrow \infty$, then $fh + f_1h_1 = O(\varphi_1)$. Analogous statements are valid for the o-relation.

If $f(x)$ is differentiable, then (1) is valid for an interval (a, b), cf. §1, (6); if $f(x)$ has an absolutely integrable derivative and approaches zero as $\xi \longrightarrow \infty$, then (1) is valid for an interval $[a, \infty]$. The last statement can be verified by means of the usual partial integration formula (Appendix 8):

$$\int_a f(x)\sin\alpha x dx = \frac{1}{\alpha}\, f(a)\cos\alpha a + \frac{1}{\alpha}\int_a f'(x)\cos\alpha x dx \ .$$

2. That (1) holds on the one hand for monotonic and on the other hand for differentiable functions is no accident. There is in fact the following connection between them. If one knows that (1) holds for monotonically decreasing functions, then it follows immediately that it also holds for monotonically increasing functions, and that it holds generally for functions which can be represented as linear combinations (with complex coefficients) of monotonic functions. We shall denote, as usual, these last functions as <u>functions of bounded variation</u>. For our purposes, we shall not need the "true" concept of bounded variation. It will be sufficient for us to show directly, that each function which has an absolutely integrable derivative is of bounded variation in the sense stated above. Since if $f_1'(x) + i\,f_2'(x)$ is absolutely integrable, $f_1'(x)$ and $f_2'(x)$ are also, we need to prove our assertion only for real valued functions. Let $f(x)$ have an integrable derivative in (a, b). Then we can set

(2) $$f(x) = f(b) + \int_x^b \frac{|f'(\xi)| - f'(\xi)}{2}\,d\xi - \int_x^b \frac{|f'(\xi)| + f'(\xi)}{2}\,d\xi = f(b) + h_1(x) - h_2(x).$$