Dynamic Asset Pricing Theory

THIRD EDITION

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For Colin

Contents

Preface

THIS BOOK IS an introduction to the theory of portfolio choice and asset pricing in multiperiod settings under uncertainty. An alternate title might be *Arbitrage, Optimality, and Equilibrium,* because the book is built around the three basic constraints on asset prices: absence of arbitrage, single-agent optimality, and market equilibrium. The most important unifying principle is that any of these three conditions implies that there are "state prices," meaning positive discount factors, one for each state and date, such that the price of any security is merely the state-price weighted sum of its future payoffs. This idea can be traced to the invention by Arrow (1953) of the general equilibrium model of security markets. Identifying the state prices is the major task at hand. Technicalities are given relatively little emphasis so as to simplify these concepts and to make plain the similarities between discrete- and continuous-time models.ricing model.

To someone who came out of graduate school in the mid-eighties, the decade spanning roughly 1969–79 seems like a golden age of dynamic asset pricing theory. Robert Merton started continuous-time financial modeling with his explicit dynamic programming solution for optimal portfolio and consumption policies. This set the stage for his 1973 general equilibrium model of security prices, another milestone. His next major contribution was his arbitrage-based proof of the option pricing formula introduced by Fisher Black and Myron Scholes in 1973, and his continual development of that approach to derivative pricing. The Black-Scholes model now seems to be, by far, the most important single breakthrough of this "golden decade," and ranks alone with the Modigliani and Miller (1958) Theorem and the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965) in its overall importance for financial theory and practice. A tremendously influential simplification of the Black-Scholes model appeared in the "binomial" option pricing model of Cox, Ross, and Rubinstein (1979), who drew on an insight of Bill Sharpe.

Working with discrete-time models, LeRoy (1973), Rubinstein (1976), and Lucas (1978) developed multiperiod extensions of the CAPM. The "Lucas model" is the "vanilla flavor" of equilibrium asset pricing models. The simplest multiperiod representation of the CAPM finally appeared in Doug Breeden's continuous-time consumption-based CAPM, published in 1979. Although not published until 1985, the Cox-Ingersoll-Ross model of the term structure of interest rates appeared in the mid-seventies and is still the premier textbook example of a continuous-time general equilibrium asset pricing model with practical applications. It also ranks as one of the key breakthroughs of that decade. Finally, extending the ideas of Cox and Ross (1976) and Ross (1978), Harrison and Kreps (1979) gave an almost definitive conceptual structure to the whole theory of dynamic security prices.

Theoretical developments in the period since 1979, with relatively few exceptions, have been a mopping-up operation. Assumptions have been weakened, there have been noteworthy extensions and illustrative models, and the various problems have become much more unified under the umbrella of the Harrison-Kreps model of equivalent martingale measures. For example, the standard approach to optimal portfolio and consumption choice in continuous-time settings has become the martingale method of Cox and Huang (1989). An essentially final version of the relationship between the absence of arbitrage and the existence of equivalent martingale measures was finally obtained by Delbaen and Schachermayer (1999).

On the applied side, markets have experienced an explosion of new valuation techniques, hedging applications, and security innovation, much of this based on the Black-Scholes and related arbitrage models. No major investment bank, for example, lacks the experts or computer technology required to implement advanced mathematical models of the term structure. Because of the wealth of new applications, there has been a significant development of special models to treat stochastic volatility, jump behavior including default, and the term structure of interest rates, along with many econometric advances designed to take advantage of the resulting improvements in richness and tractability.

Although it is difficult to predict where the theory will go next, in order to promote faster progress by people coming into the field it seems wise to have some of the basics condensed into a textbook. This book is designed to be a streamlined course text, not a research monograph. Much generality is sacrificed for expositional reasons, and there is relatively little emphasis on mathematical rigor or on the existence of general equilibrium. As its title indicates, I am treating only the theoretical side

of the story. Although it might be useful to tie the theory to the empirical side of asset pricing, we have excellent treatments of the econometric modeling of financial data, such as Campbell, Lo, and MacKinlay (1997) and Gourieroux and Jasiak (2000). I also leave out some important aspects of functioning security markets, such as asymmetric information and transactions costs. I have chosen to develop only some of the essential ideas of dynamic asset pricing, and even these are more than enough to put into one book or into a one-semester course.

Other books whose treatments overlap with some of the topics treated here include Avelleneda and Laurence (2000), Björk (1998), Dana and Jeanblanc (1998), Demange and Rochet (1992), Dewynne and Wilmott (1994), Dixit and Pindyck (1993), Dothan (1990), Duffie (1988b), Harris (1987), Huang and Litzenberger (1988), Ingersoll (1987), Jarrow (1988), Karatzas (1997), Karatzas and Shreve (1998), Lamberton and Lapeyre (1997), Magill and Quinzii (1994), Merton (1990), Musiela and Rutkowski (1997), Neftci (2000), Stokey and Lucas (1989), Willmott, Dewynne, and Howison (1993), and Wilmott, Howison, and Dewynne (1995). Each has its own aims and themes. I hope that readers will find some advantage in having yet another perspective.

A reasonable way to teach a shorter course on continuous-time asset pricing out of this book is to begin with Chapter 1 or 2 as an introduction to the basic notion of state prices and then to go directly to Chapters 5 through 11. Chapter 12, on numerical methods, could be skipped at some cost to the student's ability to implement the results. There is no direct dependence of any results in Chapters 5 through 12 on the first four chapters.

For mathematical preparation, little beyond undergraduate analysis, as in Bartle (1976), and linear algebra is assumed. Some familiarity with Royden (1968) or a similar text on functional analysis and measure theory, would also be useful. Some background in microeconomics would be useful, say Kreps (1990) or Luenberger (1995). Familiarity with probability theory at the level of Jacod and Protter (2000), for example, would also speed things along, although measure theory is not used heavily. In any case, a series of appendices supplies all of the required concepts and definitions from probability theory and stochastic calculus. Additional useful references in this regard are Brémaud (1981), Karatzas and Shreve (1988), Revuz and Yor (1991), and Protter (1990).

Students seem to learn best by doing problem exercises. Each chapter has exercises and notes to the literature. I have tried to be thorough in giving sources for results whenever possible and plead that any cases in which I have mistaken or missed sources be brought to my attention for correction. The notation and terminology throughout is fairly standard. I use $\mathbb{R}$ to denote the real line and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ for the extended real line. For any set Z and positive integer n, I use Z^n for the set of n-tuples of the form $(z_1, \ldots, z_n)$ with z_i in Z for all i. An example is $\mathbb{R}^n$. The conventions used for inequalities in any context are

- $x \geq 0$ means that x is nonnegative. For x in $\mathbb{R}^n$, this is equivalent to $x \in \mathbb{R}^n_+$;
- $x > 0$ means that x is nonnegative and not zero, but not necessarily strictly positive in all coordinates;
- $x \gg 0$ means x is strictly positive in every possible sense. The phrase "x is strictly positive" means the same thing. For x in $\mathbb{R}^n$, this is equivalent to $x \in \mathbb{R}^n_{++} \equiv \text{int}(\mathbb{R}^n_+)$.

Although warnings will be given at appropriate times, it should be kept in mind that $X = Y$ will be used to mean equality almost everywhere or almost surely, as the case may be. The same caveat applies to each of the above inequalities. A real-valued function F on an ordered set (such as $\mathbb{R}^n$) is *increasing* if $F(x) \geq F(y)$ whenever $x \geq y$ and *strictly increasing* if $F(x) > F(y)$ whenever $x > y$. When the domain and range of a function are implicitly obvious, the notation "$x \mapsto F(x)$" means the function that maps x to $F(x)$; for example, $x \mapsto x^2$ means the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x^2$. Also, while warnings appear at appropriate places, it is worth pointing out again here that, for ease of exposition, a continuous-time "process" will be defined throughout as a jointly (product) measurable function on $\Omega \times [0, T]$, where $[0, T]$ is the given time interval and $(\Omega, \mathcal{F}, P)$ is the given underlying probability space.

The first four chapters are in a discrete-time setting with a discrete set of states. This should ease the development of intuition for the models to be found in Chapters 5 through 12. The three pillars of the theory, *arbitrage, optimality,* and *equilibrium,* are developed repeatedly in different settings. Chapter 1 is the basic single-period model. Chapter 2 extends the results of Chapter 1 to many periods. Chapter 3 specializes Chapter 2 to a Markov setting and illustrates dynamic programming as an alternate solution technique. The Ho-and-Lee and Black-Derman-Toy term-structure models are included as exercises. Chapter 4 is an infinite-horizon counterpart to Chapter 3 that has become known as the *Lucas model.*

The focus of the theory is the notion of state prices, which specify the price of any security as the state-price weighted sum or expectation of the security's state-contingent dividends. In a finite-dimensional setting, there exist state prices if and only if there is no arbitrage. The same fact is true in infinite-dimensional settings under mild technical regularity conditions. Given an agent's optimal portfolio choice, a state-price vector is given by that agent's utility gradient. In an equilibrium with Pareto optimality, a state-price vector is likewise given by a representative agent's utility gradient at the economy's aggregate consumption process.

Chapters 5 through 11 develop a continuous-time version of the theory in which uncertainty is generated by Brownian motion. In Chapter 11, there is a transition to discontinuous information, that is, settings in which the conditional probability of some events does not adjust continuously with the passage of time. An example is Poisson arrival.

Chapter 5 introduces the continuous-trading model and develops the Black-Scholes partial differential equation (PDE) for arbitrage-free prices of derivative securities. The Harrison-Kreps model of equivalent martingale measures is presented in Chapter 6 in parallel with the theory of state prices in continuous time. Chapter 7 presents models of the term structure of interest rates, including the Black-Derman-Toy, Vasicek, Cox-Ingersoll-Ross, and Heath-Jarrow-Morton models, as well as extensions. Chapter 8 presents specific classes of derivative securities, such as futures, forwards, American options, and lookback options. Chapter 8 also introduces models of option pricing with stochastic volatility. The notion of an "affine" state process is used heavily in Chapters 7 and 8 for its analytical tractability. Chapter 9 is a summary of optimal continuous-time portfolio choice, using both dynamic programming and an approach involving equivalent martingale measures or state prices. Chapter 10 is a summary of security pricing in an equilibrium setting. Included are such well-known models as Breeden's consumption-based capital asset pricing model and the general equilibrium version of the Cox-Ingersoll-Ross model of the term structure of interest rates. Chapter 11 deals with the valuation of corporate securities, such as debt and equity. The chapter moves from models based on the capital structure of the corporation, in which default is defined in terms of the sufficiency of assets, to models based on an assumed process for the default arrival intensity. Chapter 12 outlines three numerical methods for calculating derivative security prices in a continuous-time setting: binomial approximation, Monte Carlo simulation of a discrete-time approximation of security prices, and finite-difference solution of the associated PDE for the asset price or the fundamental solution.

In preparing the first edition, I relied on help from many people, in addition to those mentioned above who developed this theory. In 1982, Michael Harrison gave a class at Stanford that had a major effect on my understanding and research goals. Beside me in that class was Chi-fu Huang; we learned much of this material together, becoming close friends and collaborators. I owe him a lot. I am grateful to Niko and Vana Skiadas, who treated me with overwhelming warmth and hospitality at their home on Skiathos, where parts of the first draft were written.

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For the reader's convenience, the original preface has been revised for this third edition. Significant improvements have been made in most chapters. Chapter 11, "Corporate Securities," has been added for this edition. Errors are my own responsibility, and I hope to hear of them and any other comments from readers.

Darrell Duffie

I
Discrete-Time Models

This first part of the book takes place in a discrete-time setting with a discrete set of states. This should ease the development of intuition for the models to be found in Part II. The three pillars of the theory, *arbitrage*, *optimality*, and *equilibrium*, are developed repeatedly in different settings. Chapter 1 is the basic single-period model. Chapter 2 extends the results of Chapter 1 to many periods. Chapter 3 specializes Chapter 2 to a Markov setting and illustrates dynamic programming as an alternate solution technique. The Ho-and-Lee and Black-Derman-Toy term-structure models are included as exercises. Chapter 4 is an infinite-horizon counterpart to Chapter 3 that has become known as the *Lucas model*.

The focus of the theory is the notion of state prices, which specify the price of any security as the state-price weighted sum or expectation of the security's state-contingent dividends. In a finite-dimensional setting, there exist state prices if and only if there is no arbitrage. The same fact is true in infinite-dimensional settings under mild technical regularity conditions. Given an agent's optimal portfolio choice, a state-price vector is given by that agent's utility gradient. In an equilibrium with Pareto optimality, a state-price vector is likewise given by a representative agent's utility gradient at the economy's aggregate consumption process.

1
Introduction to State Pricing

THIS CHAPTER INTRODUCES the basic ideas in a finite-state one-period setting. In many basic senses, each subsequent chapter merely repeats this one from a new perspective. The objective is a characterization of security prices in terms of "state prices," one for each state of the world. The price of a given security is simply the state-price weighted sum of its payoffs in the different states. One can treat a state price as the "shadow price," or Lagrange multiplier, for wealth contingent on a given state of the world. We obtain a characterization of state prices, first based on the absence of arbitrage, then based on the first-order conditions for optimal portfolio choice of a given agent, and finally from the first-order conditions for Pareto optimality in an equilibrium with complete markets. State prices are connected with the "beta" model for excess expected returns, a special case of which is the Capital Asset Pricing Model (CAPM). Many readers will find this chapter to be a review of standard results. In most cases, here and throughout, technical conditions are imposed that give up much generality so as to simplify the exposition.

A. Arbitrage and State Prices

Uncertainty is represented here by a finite set $\{1, \ldots, S\}$ of states, one of which will be revealed as true. The N securities are given by an $N \times S$ matrix D, with D_{ij} denoting the number of units of account paid by security i in state j. The security prices are given by some q in $\mathbb{R}^N$. A *portfolio* $\theta \in \mathbb{R}^N$ has *market value* $q \cdot \theta$ and *payoff* $D^\top \theta$ in $\mathbb{R}^S$. An *arbitrage* is a portfolio θ in $\mathbb{R}^N$ with $q \cdot \theta \leq 0$ and $D^\top \theta > 0$, or $q \cdot \theta < 0$ and $D^\top \theta \geq 0$. An arbitrage is therefore, in effect, a portfolio offering "something for nothing." Not surprisingly, it will later be shown that an arbitrage is naturally ruled out, and this gives a characterization of security prices as follows. A

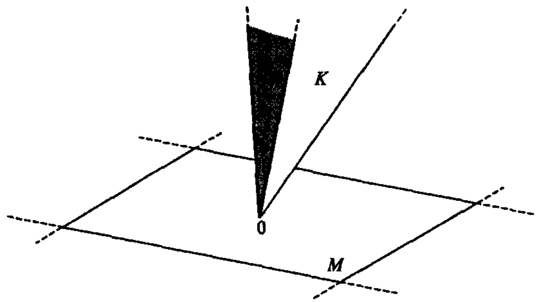

Figure 1.1. Separating a Cone from a Linear Subspace

state-price vector is a vector ψ in $\mathbb{R}^S_{++}$ with $q = D\psi$. We can think of ψ_j as the marginal cost of obtaining an additional unit of account in state j.

Theorem. *There is no arbitrage if and only if there is a state-price vector.*

Proof: The proof is an application of the Separating Hyperplane Theorem. Let $L = \mathbb{R} \times \mathbb{R}^S$ and $M = \{(-q \cdot \theta, D^\top \theta) : \theta \in \mathbb{R}^N\}$, a linear subspace of L. Let $K = \mathbb{R}_+ \times \mathbb{R}^S_+$, which is a cone (meaning that if x is in K, then λx is in K for each strictly positive scalar λ). Both K and M are closed and convex subsets of L. There is no arbitrage if and only if K and M intersect precisely at 0, as pictured in Figure 1.1.

Suppose $K \cap M = \{0\}$. The Separating Hyperplane Theorem (in a version for closed cones that is found in Appendix B) implies the existence of a nonzero linear functional $F : L \to \mathbb{R}$ such that $F(z) < F(x)$ for all z in M and nonzero x in K. Since M is a linear space, this implies that $F(z) = 0$ for all z in M and that $F(x) > 0$ for all nonzero x in K. The latter fact implies that there is some $\alpha > 0$ in $\mathbb{R}$ and $\psi \gg 0$ in $\mathbb{R}^S$ such that $F(v, c) = \alpha v + \psi \cdot c$, for any $(v, c) \in L$. This in turn implies that $-\alpha q \cdot \theta + \psi \cdot (D^\top \theta) = 0$ for all θ in $\mathbb{R}^N$. The vector ψ/α is therefore a state-price vector.

Conversely, if a state-price vector ψ exists, then for any θ, we have $q \cdot \theta = \psi^\top D^\top \theta$. Thus, when $D^\top \theta \geq 0$, we have $q \cdot \theta \geq 0$, and when $D^\top \theta > 0$, we have $q \cdot \theta > 0$, so there is no arbitrage. ∎

B. Risk-Neutral Probabilities

We can view any p in $\mathbb{R}^S_+$ with $p_1 + \cdots + p_S = 1$ as a vector of probabilities of the corresponding states. Given a state-price vector ψ for the dividend-price pair (D, q), let $\psi_0 = \psi_1 + \cdots + \psi_S$ and, for any state j, let $\hat{\psi}_j = \psi_j/\psi_0$.

We now have a vector $(\hat{\psi}_1, \ldots, \hat{\psi}_S)$ of probabilities and can write, for an arbitrary security i,

$$\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{S} \hat{\psi}_j D_{ij},$$

viewing the normalized price of the security as its expected payoff under specially chosen "risk-neutral" probabilities. If there exists a portfolio $\bar{\theta}$ with $D^\top \bar{\theta} = (1, 1, \ldots, 1)$, then $\psi_0 = \bar{\theta} \cdot q$ is the discount on riskless borrowing and, for any security i, $q_i = \psi_0 \hat{E}(D_i)$, showing any security's price to be its discounted expected payoff in this sense of artificially constructed probabilities.

C. Optimality and Asset Pricing

Suppose the dividend-price pair (D, q) is given. An *agent* is defined by a strictly increasing *utility function* $U : \mathbb{R}^S_+ \to \mathbb{R}$ and an *endowment* e in $\mathbb{R}^S_+$. This leaves the *budget-feasible set*

$$X(q, e) = \{e + D^\top \theta \in \mathbb{R}^S_+ : \theta \in \mathbb{R}^N, q \cdot \theta \leq 0\},$$

and the problem

$$\sup_{c \in X(q,e)} U(c). \tag{1}$$

We will suppose for this section that there is some portfolio θ^0 with payoff $D^\top \theta^0 > 0$. Because U is strictly increasing, the wealth constraint $q \cdot \theta \leq 0$ is then binding at an optimum. That is, if $c^* = e + D^\top \theta^*$ solves (1), then $q \cdot \theta^* = 0$.

Proposition. *If there is a solution to* (1), *then there is no arbitrage. If U is continuous and there is no arbitrage, then there is a solution to* (1).

Proof is left as an exercise.

Theorem. *Suppose that c^* is a strictly positive solution to* (1), *that U is continuously differentiable at c^*, and that the vector $\partial U(c^*)$ of partial derivatives of U at c^* is strictly positive. Then there is some scalar $\lambda > 0$ such that $\lambda \partial U(c^*)$ is a state-price vector.*

Proof: The first-order condition for optimality is that for any θ with $q \cdot \theta = 0$, the marginal utility for buying the portfolio θ is zero. This is expressed more precisely in the following way. The strict positivity of c^* implies that for any portfolio θ, there is some scalar $k > 0$ such that $c^* + \alpha D^\top \theta \geq 0$ for all α in $[-k, k]$. Let $g_\theta : [-k, k] \to \mathbb{R}$ be defined by

$$g_\theta(\alpha) = U(c^* + \alpha D^\top \theta).$$

Suppose $q \cdot \theta = 0$. The optimality of c^* implies that g_θ is maximized at $\alpha = 0$. The first-order condition for this is that $g'_\theta(0) = \partial U(c^*)^\top D^\top \theta = 0$. We can conclude that, for any θ in $\mathbb{R}^N$, if $q \cdot \theta = 0$, then $\partial U(c^*)^\top D^\top \theta = 0$. From this, there is some scalar μ such that $\partial U(c^*)^\top D^\top = \mu q$.

By assumption, there is some portfolio θ^0 with $D^\top \theta^0 > 0$. From the existence of a solution to (1), there is no arbitrage, implying that $q \cdot \theta^0 > 0$. We have

$$\mu q \cdot \theta^0 = \partial U(c^*)^\top D^\top \theta^0 > 0.$$

Thus $\mu > 0$. We let $\lambda = 1/\mu$, obtaining

$$q = \lambda D \partial U(c^*), \tag{2}$$

implying that $\lambda \partial U(c^*)$ is a state-price vector. ∎

Although we have assumed that U is strictly increasing, this does not necessarily mean that $\partial U(c^*) \gg 0$. If U is concave and strictly increasing, however, it is always true that $\partial U(c^*) \gg 0$.

Corollary. *Suppose U is concave and differentiable at some $c^* = e + D^\top \theta^* \gg 0$, with $q \cdot \theta^* = 0$. Then c^* is optimal if and only if $\lambda \partial U(c^*)$ is a state-price vector for some scalar $\lambda > 0$.*

This follows from the sufficiency of the first-order optimality conditions for concave objective functions. The idea is illustrated in Figure 1.2. In that figure, there are only two states, and a state-price vector is a suitably normalized nonzero positive vector orthogonal to the set $B = \{D^\top \theta : q \cdot \theta = 0\}$ of budget-neutral consumption adjustments. The first-order condition for optimality of c^* is that movement in any feasible direction away from c^* has negative or zero marginal utility, which is equivalent to the statement that the budget-neutral set is tangent at c^* to the preferred set $\{c : U(c) \geq U(c^*)\}$, as shown in the figure. This is equivalent to the statement that $\partial U(c^*)$ is orthogonal to B, consistent with the last corollary. Figure 1.3 illustrates a strictly suboptimal consumption choice c, at which the derivative vector $\partial U(c)$ is not co-linear with the state-price vector ψ.

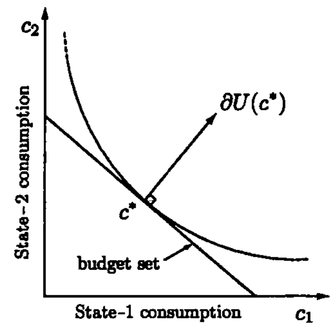

Figure 1.2. First-Order Conditions for Optimal Consumption Choice

We consider the special case of an *expected utility* function U, defined by a given vector p of probabilities and by some $u : \mathbb{R}_+ \to \mathbb{R}$ according to

$$U(c) = E[u(c)] \equiv \sum_{j=1}^{S} p_j u(c_j). \tag{3}$$

For $c \gg 0$, if u is differentiable, then $\partial U(c)_j = p_j u'(c_j)$. For this expected utility function, (2) therefore applies if and only if

$$q = \lambda E[Du'(c^*)], \tag{4}$$

with the obvious notational convention. As we saw in Section B, one can also write (2) or (4), with the "risk-neutral" probability $\hat{\psi}_j = u'(c^*_j)p_j / E[u'(c^*)]$, in the form

$$\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{S} D_{ij}\hat{\psi}_j, \qquad 1 \leq i \leq N. \tag{5}$$

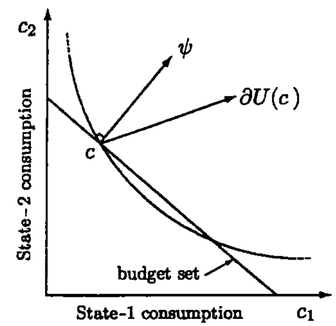

Figure 1.3. A Strictly Suboptimal Consumption Choice

D. Efficiency and Complete Markets

Suppose there are m agents, defined as in Section C by strictly increasing utility functions $U_1, \ldots, U_m$ and by endowments $e^1, \ldots, e^m$. An *equilibrium* for the *economy* $[(U_i, e^i), D]$ is a collection $(\theta^1, \ldots, \theta^m, q)$ such that, given the security-price vector q, for each agent i, θ^i solves $\sup_\theta U_i(e^i + D^\top \theta)$ subject to $q \cdot \theta \leq 0$, and such that $\sum_{i=1}^m \theta^i = 0$. The existence of equilibrium is treated in the exercises and in sources cited in the Notes.

With $\text{span}(D) \equiv \{D^\top \theta : \theta \in \mathbb{R}^N\}$ denoting the set of possible portfolio payoffs, markets are *complete* if $\text{span}(D) = \mathbb{R}^S$, and are otherwise *incomplete*.

Let $e = e^1 + \cdots + e^m$ denote the aggregate endowment. A consumption *allocation* $(c^1, \ldots, c^m)$ in $(\mathbb{R}^S_+)^m$ is *feasible* if $c^1 + \cdots + c^m \leq e$. A feasible allocation $(c^1, \ldots, c^m)$ is *Pareto optimal* if there is no feasible allocation $(\hat{c}^1, \ldots, \hat{c}^m)$ with $U_i(\hat{c}^i) \geq U_i(c^i)$ for all i and with $U_i(\hat{c}^i) > U_i(c^i)$ for some i. Complete markets and the Pareto optimality of equilibrium allocations are almost equivalent properties of any economy.

Proposition. *Suppose markets are complete and $(\theta^1, \ldots, \theta^m, q)$ is an equilibrium. Then the associated equilibrium allocation is Pareto optimal.*

This is sometimes known as *The First Welfare Theorem.* The proof, requiring only the strict monotonicity of utilities, is left as an exercise. We have established the sufficiency of complete markets for Pareto optimality. The necessity of complete markets for the Pareto optimality of equilibrium allocations does not always follow. For example, if the initial endowment allocation $(e^1, \ldots, e^m)$ happens by chance to be Pareto optimal, then any equilibrium allocation is also Pareto optimal, regardless of the span of securities. It would be unusual, however, for the initial endowment to be Pareto optimal. Although beyond the scope of this book, it can be shown that with incomplete markets and under natural assumptions on utility, for almost every endowment, the equilibrium allocation is not Pareto optimal.

E. Optimality and Representative Agents

Aside from its allocational implications, Pareto optimality is also a convenient property for the purpose of security pricing. In order to see this, consider, for each vector $\lambda \in \mathbb{R}^m_+$ of "agent weights," the utility function $U_\lambda : \mathbb{R}^S_+ \to \mathbb{R}$ defined by

$$U_\lambda(x) = \sup_{(c^1, \ldots, c^m)} \sum_{i=1}^m \lambda_i U_i(c^i) \qquad \text{subject to } c^1 + \cdots + c^m \leq x. \tag{6}$$

Lemma. *Suppose that, for all i, U_i is concave. An allocation $(c^1, \ldots, c^m)$ that is feasible is Pareto optimal if and only if there is some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(c^1, \ldots, c^m)$ solves (6) at $x = e = c^1 + \cdots + c^m$.*

Proof: Suppose that $(c^1, \ldots, c^m)$ is Pareto optimal. For any allocation x, let $U(x) = (U_1(x^1), \ldots, U_m(x^m))$. Next, let

$$\mathcal{U} = \{U(x) - U(c) - z : x \in \mathcal{A},\ z \in \mathbb{R}^m_+\} \subset \mathbb{R}^m,$$

where $\mathcal{A}$ is the set of feasible allocations. Let $J = \{y \in \mathbb{R}^m_+ : y \neq 0\}$. Since $\mathcal{U}$ is convex (by the concavity of utility functions) and $J \cap \mathcal{U}$ is empty (by Pareto optimality), the Separating Hyperplane Theorem (Appendix B) implies that there is a nonzero vector λ in $\mathbb{R}^m$ such that $\lambda \cdot y \leq \lambda \cdot z$ for each y in $\mathcal{U}$ and each z in J. Since $0 \in \mathcal{U}$, we know that $\lambda \geq 0$, proving the first part of the result. The second part is easy to show as an exercise. ∎

Proposition. *Suppose that for all i, U_i is concave. Suppose that markets are complete and that $(\theta^1, \ldots, \theta^m, q)$ is an equilibrium. Then there exists some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(0, q)$ is a (no-trade) equilibrium for the single-agent economy $[(U_\lambda, e), D]$ defined by (6). Moreover, the equilibrium consumption allocation $(c^1, \ldots, c^m)$ solves the allocation problem (6) at the aggregate endowment. That is, $U_\lambda(e) = \sum_i \lambda_i U_i(c^i)$.*

Proof: Since there is an equilibrium, there is no arbitrage, and therefore there is a state-price vector ψ. Since markets are complete, this implies that the problem of any agent i can be reduced to

$$\sup_{c \in \mathbb{R}^S_+} U_i(c) \quad \text{subject to } \psi \cdot c \leq \psi \cdot e^i.$$

We can assume that e^i is not zero, for otherwise $c^i = 0$ and agent i can be eliminated from the problem without loss of generality. By the Saddle Point Theorem of Appendix B, there is a Lagrange multiplier $\alpha_i \geq 0$ such that c^i solves the problem

$$\sup_{c \in \mathbb{R}^S_+} U_i(c) - \alpha_i(\psi \cdot c - \psi \cdot e^i).$$

(The Slater condition is satisfied since e^i is not zero and $\psi \gg 0$.) Since U_i is strictly increasing, $\alpha_i > 0$. Let $\lambda_i = 1/\alpha_i$. For any feasible allocation $(x^1, \ldots, x^m)$, we have

$$\sum_{i=1}^m \lambda_i U_i(c^i) = \sum_{i=1}^m [\lambda_i U_i(c^i) - \lambda_i \alpha_i(\psi \cdot c^i - \psi \cdot e^i)]$$

$$\geq \sum_{i=1}^{m} \lambda_i [U_i(x^i) - \alpha_i(\psi \cdot x^i - \psi \cdot e^i)]$$

$$= \sum_{i=1}^{m} \lambda_i U_i(x^i) - \psi \cdot \sum_{i=1}^{m} (x^i - e^i)$$

$$\geq \sum_{i=1}^{m} \lambda_i U_i(x^i).$$

This shows that $(c^1, \ldots, c^m)$ solves the allocation problem (6). We must also show that no trade is optimal for the single agent with utility function U_λ and endowment e. If not, there is some x in $\mathbb{R}^S_+$ such that $U_\lambda(x) > U_\lambda(e)$ and $\psi \cdot x \leq \psi \cdot e$. By the definition of U_λ, this would imply the existence of an allocation $(x^1, \ldots, x^m)$, not necessarily feasible, such that $\sum_i \lambda_i U_i(x^i) > \sum_i \lambda_i U_i(c^i)$ and

$$\sum_i \lambda_i \alpha_i \psi \cdot x^i = \psi \cdot x \leq \psi \cdot e = \sum_i \lambda_i \alpha_i \psi \cdot c^i.$$

Putting these two inequalities together, we have

$$\sum_{i=1}^{m} \lambda_i [U_i(x^i) - \alpha_i \psi \cdot (x^i - e^i)] > \sum_{i=1}^{m} \lambda_i [U_i(c^i) - \alpha_i \psi \cdot (c^i - e^i)],$$

which contradicts the fact that, for each agent i, (c^i, α_i) is a saddle point for that agent's problem. ∎

Corollary 1. *If, moreover,* $e \gg 0$ *and* U_λ *is continuously differentiable at* e*, then* λ *can be chosen so that* $\partial U_\lambda(e)$ *is a state-price vector, meaning*

$$q = D \partial U_\lambda(e). \tag{7}$$

The differentiability of U_λ at e is implied by the differentiability, for some agent i, of U_i at c^i. (See Exercise 10(C).)

Corollary 2. *Suppose there is a fixed vector* p *of state probabilities such that, for all* i, $U_i(c) = E[u_i(c)] \equiv \sum_{j=1}^{S} p_j u_i(c_j)$, *for some* $u_i(\cdot)$. *Then* $U_\lambda(c) = E[u_\lambda(c)]$, *where, for each* y *in* $\mathbb{R}_+$,

$$u_\lambda(y) = \max_{x \in \mathbb{R}^m_+} \sum_{i=1}^{m} \lambda_i u_i(x_i) \quad \text{subject to } x_1 + \cdots + x_m \leq y.$$

In this case, (7) is equivalent to $q = E[Du'_\lambda(e)]$.

Extensions of this *representative-agent* asset pricing formula will crop up frequently in later chapters.

F. State-Price Beta Models

We fix a vector $p \gg 0$ in $\mathbb{R}^S$ of probabilities for this section, and for any x in $\mathbb{R}^S$ we write $E(x) = p_1 x_1 + \cdots + p_S x_S$. For any x and π in $\mathbb{R}^S$, we take $x\pi$ to be the vector $(x_1\pi_1, \ldots, x_S\pi_S)$. The following version of the *Riesz Representation Theorem* can be shown as an exercise.

Lemma. *Suppose* $F : \mathbb{R}^S \to \mathbb{R}$ *is linear. Then there is a unique* π *in* $\mathbb{R}^S$ *such that, for all* x *in* $\mathbb{R}^S$, *we have* $F(x) = E(\pi x)$. *Moreover,* F *is strictly increasing if and only if* $\pi \gg 0$.

Corollary. *A dividend-price pair* (D, q) *admits no arbitrage if and only if there is some* $\pi \gg 0$ *in* $\mathbb{R}^S$ *such that* $q = E(D\pi)$.

Proof: Given a state-price vector ψ, let $\pi_s = \psi_s / p_s$. Conversely, if π has the assumed property, then $\psi_s = p_s \pi_s$ defines a state-price vector ψ. ∎

Given (D, q), we refer to any vector π given by this result as a *state-price deflator.* (The terms *state-price density* and *state-price kernel* are often used synonymously with state-price deflator.) For example, the representative-agent pricing model of Corollary 2 of Section E shows that we can take $\pi_s = u'_\lambda(e_s)$.

For any x and y in $\mathbb{R}^S$, the *covariance* $\text{cov}(x, y) \equiv E(xy) - E(x)E(y)$ is a measure of covariation between x and y that is useful in asset pricing applications. For any such x and y with $\text{var}(y) \equiv \text{cov}(y, y) \neq 0$, we can always represent x in the form $x = \alpha + \beta y + \epsilon$, where $\beta = \text{cov}(y, x)/\text{var}(y)$, where $\text{cov}(y, \epsilon) = E(\epsilon) = 0$, and where α is a scalar. This *linear regression* of x on y is uniquely defined. The coefficient β is called the associated *regression coefficient.*

Suppose (D, q) admits no arbitrage. For any portfolio θ with $q \cdot \theta \neq 0$, the *return* on θ is the vector R^θ in $\mathbb{R}^S$ defined by $R^\theta_s = (D^\top \theta)_s / q \cdot \theta$. Fixing a state-price deflator π, for any such portfolio θ, we have $E(\pi R^\theta) = 1$. Suppose there is a *riskless portfolio,* meaning some portfolio θ with constant return R^0. We then call R^0 the *riskless return.* A bit of algebra shows that for any portfolio θ with a return, we have

$$E(R^\theta) - R^0 = -\frac{\text{cov}(R^\theta, \pi)}{E(\pi)}.$$

Thus, covariation with π has a negative effect on expected return, as one might expect from the interpretation of state prices as shadow prices for wealth.

The *correlation* between any x and y in $\mathbb{R}^S$ is zero if either has zero variance, and is otherwise defined by

$$\text{corr}(x, y) = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x)\,\text{var}(y)}}.$$

There is always a portfolio θ^* solving the problem

$$\sup_\theta \ \text{corr}(D^\top \theta, \pi). \tag{8}$$

If there is such a portfolio θ^* with a return R^* having nonzero variance, then it can be shown as an exercise that, for any return R^θ,

$$E(R^\theta) - R^0 = \beta_\theta[E(R^*) - R^0], \tag{9}$$

where

$$\beta_\theta = \frac{\text{cov}(R^*, R^\theta)}{\text{var}(R^*)}.$$

If markets are complete, then R^* is of course perfectly correlated with the state-price deflator.

Formula (9) is a *state-price beta model*, showing excess expected returns on portfolios to be proportional to the excess return on a portfolio having maximal correlation with a state-price deflator, where the constant of proportionality is the associated regression coefficient. The formula can be extended to the case in which there is no riskless return. Another exercise carries this idea, under additional assumptions, to the *Capital Asset Pricing Model*, or *CAPM*.

Exercises

1.1 The dividend-price pair (D, q) of Section A is defined to be *weakly arbitrage-free* if $q \cdot \theta \geq 0$ whenever $D^\top \theta \geq 0$. Show that (D, q) is weakly arbitrage-free if and only if there exist ("weak" state prices) $\psi \in \mathbb{R}^S_+$ such that $q = D\psi$. This fact is known as *Farkas's Lemma*.

1.2 Prove the assertion in Section A that (D, q) is arbitrage-free if and only if there exists some $\psi \in \mathbb{R}^S_{++}$ such that $q = D\psi$. Instead of following the proof given in Section A, use the following result, sometimes known as the *Theorem of the Alternative*.

Stiemke's Lemma. *Suppose A is an $m \times n$ matrix. Then one and only one of the following is true:*

(a) *There exists x in $\mathbb{R}^n_{++}$ with $Ax = 0$.*
(b) *There exists y in $\mathbb{R}^m$ with $y^\top A > 0$.*

1.3 Show, for $U(c) \equiv E[u(c)]$ as defined by (3), that (2) is equivalent to (4).

1.4 Prove the existence of an equilibrium as defined in Section D under these assumptions: There exists some portfolio θ with payoff $D^\top \theta > 0$ and, for all i, $e^i \gg 0$ and U_i is continuous, strictly concave, and strictly increasing. This is a demanding exercise, and calls for the following general result.

Kakutani's Fixed Point Theorem. *Suppose Z is a nonempty convex compact subset of $\mathbb{R}^n$, and for each x in Z, $\varphi(x)$ is a nonempty convex compact subset of Z. Suppose also that $\{(x, y) \in Z \times Z : x \in \varphi(y)\}$ is closed. Then there exists x^* in Z such that $x^* \in \varphi(x^*)$.*

1.5 Prove Proposition D. Hint: The maintained assumption of strict monotonicity of $U_i(\cdot)$ should be used.

1.6 Suppose that the endowment allocation $(e^1, \ldots, e^m)$ is Pareto optimal.

(A) Show, as claimed in Section D, that any equilibrium allocation is Pareto optimal.

(B) Suppose that there is some portfolio θ with $D^\top \theta > 0$ and, for all i, that U_i is concave and $e^i \gg 0$. Show that $(e^1, \ldots, e^m)$ is itself an equilibrium allocation.

1.7 Prove Proposition C. Hint: A continuous real-valued function on a compact set has a maximum.

1.8 Prove Corollary 1 of Proposition E.

1.9 Prove Corollary 2 of Proposition E.

1.10 Suppose, in addition to the assumptions of Proposition E, that

(a) $e = e^1 + \cdots + e^m$ is in $\mathbb{R}^S_{++}$;
(b) for all i, U_i is concave and twice continuously differentiable in $\mathbb{R}^S_{++}$;
(c) for all i, c^i is in $\mathbb{R}^S_{++}$ and the Hessian matrix $\partial^2 U(c^i)$, which is negative semi-definite by concavity, is in fact negative definite.

Property (c) can be replaced with the assumption of *regular preferences*, as defined in a source cited in the Notes.

(A) Show that the assumption that U_λ is continuously differentiable at e is justified and, moreover, that for each i there is a scalar $\gamma_i > 0$ such that $\partial U_\lambda(e) = \gamma_i \partial U_i(c^i)$. (This co-linearity is known as "equal marginal rates of substitution," a property of any Pareto optimal allocation.) Hint: Use the following:

Implicit Function Theorem. *Suppose for given m and n that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is C^k (k times continuously differentiable) for some $k \geq 1$. Suppose also that the $n \times n$ matrix $\partial_y f(\bar{x}, \bar{y})$ of partial derivatives of f with respect to its second argument is nonsingular at some $(\bar{x}, \bar{y})$. If $f(\bar{x}, \bar{y}) = 0$, then there exist scalars $\epsilon > 0$ and $\delta > 0$ and a C^k function $Z : \mathbb{R}^m \to \mathbb{R}^n$ such that if $\|x - a\| < \epsilon$, then $f[x, Z(x)] = 0$ and $\|Z(x) - b\| < \delta$.*

(B) Show that the negative-definite part of condition (c) is satisfied if $e \gg 0$ and, for all i, U_i is an expected utility function of the form $U_i(c) = E[u_i(c)]$, where u_i is strictly concave with an unbounded derivative on $(0, \infty)$.

(C) Obtain the result of part (A) without assuming the existence of second derivatives of the utilities. (You would therefore not exploit the Hessian matrix or Implicit Function Theorem.) As the first (and main) step, show the following. Given a concave function $f : \mathbb{R}^S_+ \to \mathbb{R}$, the *superdifferential* of f at some x in $\mathbb{R}^S_+$ is

$$\partial f(x) = \{z \in \mathbb{R}^S : f(y) \leq f(x) + z \cdot (y - x), \quad y \in \mathbb{R}^S_+\}.$$

For any feasible allocation $(c^1, \ldots, c^m)$ and $\lambda \in \mathbb{R}^m_+$ satisfying $U_\lambda(e) = \sum_i \lambda_i U_i(c^i)$,

$$\partial U_\lambda(e) = \bigcap_{i=1}^{m} \lambda_i \partial U_i(c_i).$$

1.11 (Binomial Option Pricing). As an application of the results in Section A, consider the following two-state ($S = 2$) option-pricing problem. There are $N = 3$ securities:

(a) a stock, with initial price $q_1 > 0$ and dividend $D_{11} = Gq_1$ in state 1 and dividend $D_{12} = Bq_1$ in state 2, where $G > B > 0$ are the "good" and "bad" gross returns, respectively;
(b) a riskless bond, with initial price $q_2 > 0$ and dividend $D_{21} = D_{22} = Rq_2$ in both states (that is, R is the riskless return and R^{-1} is the discount);
(c) a *call option* on the stock, with initial price $q_3 = C$ and dividend $D_{3j} = (D_{1j} - K)^+ \equiv \max(D_{1j} - K, 0)$ for both states $j = 1$ and $j = 2$, where $K \geq 0$ is the *exercise price* of the option. (The call option gives its holder the right, but not the obligation, to pay K for the stock, with dividend, after the state is revealed.)

(A) Show necessary and sufficient conditions on G, B, and R for the absence of arbitrage involving only the stock and bond.

(B) Assuming no arbitrage for the three securities, calculate the call-option price C explicitly in terms of q_1, G, R, B, and K. Find the state-price probabilities $\hat{\psi}_1$ and $\hat{\psi}_2$ referred to in Section B in terms of G, B, and R, and show that $C = R^{-1}\hat{E}(D_3)$, where $\hat{E}$ denotes expectation with respect to $(\hat{\psi}_1, \hat{\psi}_2)$.

1.12 (CAPM). In the setting of Section D, suppose $(c^1, \ldots, c^m)$ is a strictly positive equilibrium consumption allocation. For any agent i, suppose utility is of the expected-utility form $U_i(c) = E[u_i(c)]$. For any agent i, suppose there are fixed positive constants $\bar{c}$ and b_i such that, for any state j, we have $c^i_j < \bar{c}$ and $u_i(x) = x - b_i x^2$ for all $x \leq \bar{c}$.

(A) In the context of Corollary 2 of Section E, show that $u'_\lambda(e) = k - Ke$ for some positive constants k and K. From this, derive the CAPM

$$q = AE(D) - B\,\mathrm{cov}(D, e), \qquad (10)$$

for positive constants A and B, where $\mathrm{cov}(D, e) \in \mathbb{R}^N$ is the vector of covariances between the security dividends and the aggregate endowment.

Suppose for a given portfolio θ that each of the following is well defined:

- the return $R^\theta \equiv D^\top\theta / q \cdot \theta$;
- the return R^M on a portfolio M with payoff $D^\top M = e$;
- the return R^0 on a portfolio θ^0 with $\mathrm{cov}(D^\top\theta^0, e) = 0$;
- $\beta_\theta = \mathrm{cov}(R^\theta, R^M)/\mathrm{var}(R^M)$.

The return R^M is sometimes called the *market return.* The return R^0 is called the *zero-beta return* and is the return on a riskless bond if one exists. Prove the "beta" form of the CAPM

$$E(R^\theta - R^0) = \beta_\theta E(R^M - R^0). \qquad (11)$$

(B) Part (A) relies on the completeness of markets. Without any such assumption, but assuming that the equilibrium allocation $(c^1, \ldots, c^m)$ is strictly positive, show that the same beta form (11) applies, provided we extend the definition of the market return R^M to be the return on any portfolio solving

$$\sup_{\theta \in \mathbb{R}^N} \mathrm{corr}(R^\theta, e). \qquad (12)$$

For complete markets, $\mathrm{corr}(R^M, e) = 1$, so the result of part (A) is a special case.

(C) The CAPM applies essentially as stated without the quadratic expected-utility assumption provided that each agent i is *strictly variance-averse,* in that $U_i(x) > U_i(y)$ whenever $E(x) = E(y)$ and $\mathrm{var}(x) < \mathrm{var}(y)$. Formalize this statement by providing a reasonable set of supporting technical conditions.

We remark that a common alternative formulation of the CAPM allows security portfolios in initial endowments $\hat{\theta}^1, \ldots, \hat{\theta}^m$ with $\sum_{i=1}^m \hat{\theta}^i_j = 1$ for all j. In this case, with the total endowment e redefined by $e = \sum_{i=1}^m (e^i + D^\top\hat{\theta}^i)$, the same CAPM (11) applies. If $e^i = 0$ for all i, then even in incomplete markets, $\mathrm{corr}(R^M, e) = 1$, since (12) is solved by $\theta = (1, 1, \ldots, 1)$. The Notes provide references.

1.13 An *Arrow-Debreu equilibrium* for $[(U_i, e^i), D]$ is a nonzero vector ψ in $\mathbb{R}^S_+$ and a feasible consumption allocation $(c^1, \ldots, c^m)$ such that for each i, c^i solves $\sup_c U_i(c)$ subject to $\psi \cdot c^i \leq \psi \cdot e^i$. Suppose that markets are complete, in that $\mathrm{span}(D) = \mathbb{R}^S$. Show that $(c^1, \ldots, c^m)$ is an Arrow-Debreu consumption allocation if and only if it is an equilibrium consumption allocation in the sense of Section D.

1.14 Suppose (D, q) admits no arbitrage. Show that there is a unique state-price vector if and only if markets are complete.

1.15 (Aggregation). For the "representative-agent" problem (6), suppose for all i that $U_i(c) = E[u(c)]$, where $u(c) = c^\gamma/\gamma$ for some nonzero scalar $\gamma < 1$.

(A) Show, for any nonzero agent weight vector $\lambda \in \mathbb{R}^m_+$, that $U_\lambda(c) = E[kc^\gamma/\gamma]$ for some scalar $k > 0$ and that (6) is solved by $c^i = k_i x$ for some scalar $k_i \geq 0$ that is nonzero if and only if λ_i is nonzero.