THE PHILOSOPHY OF QUANTUM MECHANICS

The Interpretations of Quantum Mechanics in Historical Perspective

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PREFACE

Never in the history of science has there been a theory which has had such a profound impact on human thinking as quantum mechanics; nor has there been a theory which scored such spectacular successes in the prediction of such an enormous variety of phenomena (atomic physics, solid state physics, chemistry, etc.). Furthermore, for all that is known today, quantum mechanics is the only consistent theory of elementary processes.

Thus although quantum mechanics calls for a drastic revision of the very foundations of traditional physics and epistemology, its mathematical apparatus or, more generally, its abstract *formalism* seems to be firmly established. In fact, no other formalism of a radically different structure has ever been generally accepted as an alternative. The *interpretation* of this formalism, however, is today, almost half a century after the advent of the theory, still an issue of unprecedented dissension. In fact, it is by far the most controversial problem of current research in the foundations of physics and divides the community of physicists and philosophers of science into numerous opposing "schools of thought."

In spite of its importance for physics and philosophy alike, the interpretative problem of quantum mechanics has rarely, if ever, been studied *sine ira et studio* from a general historical point of view. The numerous essays and monographs published on this subject are usually confined to specific aspects in defense of a particular view. No comprehensive scholarly analysis of the problem in its generality and historical perspective has heretofore appeared. The present historico-critical study is designed to fill this lacuna.

The book is intended to serve two additional purposes.

Since the book is not merely a chronological catalogue of the various interpretations of quantum mechanics but is concerned primarily with the analysis of their conceptual backgrounds, philosophical implications, and interrelations, it may also serve as a general introduction to the study of the logical foundations and philosophy of quantum mechanics. Although indispensable for a deeper understanding of modern theoretical physics, this subject is seldom given sufficient consideration in the usual textbooks and lecture courses on the theory. The historical approach, moreover, has the didactical advantage of facilitating such a study for the uninitiated reader.

Finally, because of its detailed documentation the book may also be used as a guide to the literature of the subject. Great care has been taken to provide accurate and up-to-date references to the international literature on the topics discussed. The reader should find it easy to pursue any specific detail in which he happens to be interested. To make the book self-contained and understandable not only to the specialist but also to the general reader familiar with the rudiments of quantum physics, proofs of all the theorems which are of decisive importance for the interpretative problem are given either in detail or in outline. In addition, much of the material which is required to understand the text but is usually not included in courses on quantum mechanics is explained either in full or at least to such an extent that no difficulties should arise in following the arguments. In particular, for the convenience of the reader the essentials of lattice theory, a subject seldom studied by physicists but indispensable for a comprehension of quantum logic and related topics, are summarized in an appendix at the end of the book. Concise as it is, this summary contains all prerequisites to prove the theorems referred to in the text.

If the reader is interested only in the philosophical aspects of the subject, he may well omit some of the more technical and mathematical sections of the text and yet be able to follow the main argument without serious loss of continuity.

The notation, made uniform as much as possible, is always explained in the text. To keep the book to manageable length, repetitions of footnotes are avoided. To this end, an abbreviation like footnote 2 in Chapter 3, "Ref. 2-1 (1969, p. 105; 1971, p. 73)," is meant to refer to page 105 of the 1969 publication and to page 73 of the 1971 publication mentioned in footnote 1 of Chapter 2. In references to the same chapter the chapter number is omitted. A similar notation is adopted for references to mathematical equations.

The book has its origin in lecture notes for a graduate course on the history and philosophy of modern physics which I gave in 1968 at Columbia University (New York). The first four chapters of the book were written during my visits to the Minnesota Center for Philosophy of Science (Minneapolis), the Max-Planck-Institute (Munich and Starnberg), and the Niels Bohr Institute (Copenhagen). Chapter 5 is based on a paper which I read in 1971 at Lomonosov State University (Moscow) on the occasion of the XIIIth International Congress of the History of Science. The subsequent two chapters are expanded versions of talks which I gave in 1972 and 1973 at the International School of Physics "Enrico Fermi" at

Varenna (Italy), at the University of Florence, and at the Universities of Amsterdam. Chapter 8 was written during my visit to the Universities of Berlin, Göttingen, Hamburg, and Marburg. The last three chapters were completed at Simon Fraser University (British Columbia, Canada), the University of Alberta (Edmonton, Canada), and Reed College (Portland, Oregon) where I served in 1973 as Andrew Mellon Distinguished Visiting Professor.

These lecture engagements enabled me to consult many libraries and archives and, more important, to establish personal contact with the leading quantum theorists of our time. Disregarding what the pragmatic humanist F. C. S. Schiller once called "the curious etiquette which apparently taboos the asking of questions about a philosopher's meaning while he is alive," I unscrupulously interrogated many prominent authorities about numerous details of their work in the foundations of quantum mechanics. Their readiness and frankness in answering my questions enabled me to obtain much of the information first hand, an opportunity invaluable for one who works in the current history of physics.

I thus owe a great debt of gratitude. Most influential on my view of the role of philosophy in physics were the discussions with, and writings of, Professors Herbert Feigl, Paul K. Feyerabend, Henry Margenau, Ernest Nagel, and Wolfgang Stegmüller. As to physics proper, I wish to express my gratitude to Professors Louis de Broglie, Paul A. M. Dirac, Tsung-Dao Lee, and Eugene P. Wigner for having read parts of the manuscript or having given me the privilege of discussing with them numerous subjects dealt with in the book. I would also like to acknowledge my indebtedness to Professors Leslie E. Ballentine and David Bohm for having read substantial parts of the typescript, and to Professors Friedrich Bopp, Jeffrey Bub, Richard Friedberg, Kurt Hübner, Friedrich Hund, Josef M. Jauch, Pascual Jordan, Gerhart Lüders, Peter Mittelstaedt, Wilhelm Ochs, Rudolf Peierls, Constantin Piron, Mauritius Renninger, Nathan Rosen, Léon Rosenfeld, Mendel Sachs, Erhard Scheibe, Abner Shimony, Georg Süssmann, and Carl Friedrich von Weizsäcker for their patience in discussing with me in stimulating conversations many aspects of their work. I am also grateful to Professors Assène B. Datzeff, John Stewart Bell, Dimitrii I. Blokhintsev, Wolfgang Büchel, Hilbrand J. Groenewold, Grete Henry-Hermann, Banesh Hoffmann, Edwin C. Kemble, Alfred Landé, Sir Karl R. Popper, and Martin Strauss as well as to Dr. Hugh Everett III, Mrs. Edith London, and Mrs. (Polly) Boris Podolsky for their cooperation in providing epistolary information. Finally, I wish to thank my colleagues Professors Marshall Luban and Paul Gluck for their critical reading of the typescript of the book.

Needless to say, the responsibility for any errors or misinterpretations rests entirely upon me.

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1.1. THE FORMALISM

The purpose of the first part of this introductory chapter is to present a brief outline of the mathematical formalism of nonrelativistic quantum mechanics of systems with a finite number of degrees of freedom. This formalism, as we have shown elsewhere,[1] was the outcome of a complicated conceptual process of trial and error and it is hardly an overstatement to say that it preceded its own interpretation, a development almost unique in the history of physical science. Although the reader is assumed to be acquainted with this formalism, its essential features will be reviewed, without regard to mathematical subtleties, to introduce the substance and terminology needed for discussion of the various interpretations.

Like other physical theories, quantum mechanics can be formalized in terms of several axiomatic formulations. The historically most influential and hence for the history of the interpretations most important formalism was proposed in the late 1920s by John von Neumann and expounded in his classic treatise on the mathematical foundations of quantum mechanics.[2]

In recent years a number of excellent texts[3] have been published which discuss and elaborate von Neumann's formalism and to which the reader is referred for further details.

Von Neumann's idea to formulate quantum mechanics as an operator calculus in Hilbert space was undoubtedly one of the great innovations in modern mathematical physics.[4]

[1] M. Jammer, *The Conceptual Development of Quantum Mechanics* (McGraw-Hill, New York, 1966, 1968, 1973): *Ryōshi Riki-gaku Shi* (Tokyo Tosho, Tokyo, 1974).

[2] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932, 1969; Dover, New York, 1943); *Les Fondements Mathématiques de la Mécanique Quantique* (Alcan, Paris, 1946); *Fundamentos Matemáticos de la Mecánica Cuántica* (Instituto Jorge Juan, Madrid, 1949); *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1955); *Matematičeskije Osnovi Kvantovoj Mekhaniki* (Nauka, Moscow, 1964).

[3] G. Fano, *Metodi Matematici della Meccanica Quantistica* (Zanichelli, Bologna, 1967); *Mathematical Methods of Quantum Mechanics* (McGraw-Hill, New York, 1971). B. Sz.-Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes* (Springer, Berlin, Heidelberg, New York, 1967); J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1968); B. A. Lengyel, "Functional analysis for quantum theorists," *Advances in Quantum Chemistry* **1968**, 1–82; J. L. Soulé, *Linear Operators in Hilbert Space* (Gordon and Breach, New York, 1968); T. F. Jordan, *Linear Operators for Quantum Mechanics* (Wiley, New York, 1969); E. Prugovečki, *Quantum Mechanics in Hilbert Space* (Academic Press, New York, London, 1971).

[4] For the history of the mathematical background of this discovery see Ref. 1 and M. Bernkopf, "The development of function spaces with particular reference to their origins in integral equation theory," *Archive for History of Exact Sciences* 3, 1–96 (1966); "A history of infinite matrices," *ibid.*, 4, 308–358 (1968); E. E. Kramer, *The Nature and Growth of Modern*

A *Hilbert space* $\mathcal{H}$, as abstractly defined by von Neumann, is a linear strictly positive inner product space (generally over the field $\mathcal{F}$ of complex numbers) which is complete with respect to the metric generated by the inner product and which is separable. Its elements are called *vectors*, usually denoted by $\psi,\varphi,\ldots$, and their inner or scalar product is denoted by (φ,ψ), whereas the elements of $\mathcal{F}$ are called *scalars* and usually denoted by $a,b,\ldots$. In his work on linear integral equations (1904–1910) David Hilbert had studied two realizations of such a space, the Lebesgue space $\mathcal{L}^2$ of (classes of) all complex-valued Lebesgue measurable square-integrable functions on an interval of the real line R (or R itself), and the space l^2 of sequences of complex numbers, the sum of whose absolute squares converges. Impressed by the fact that by virtue of the Riesz-Fischer theorem these two spaces can be shown to be isomorphic (and isometric) and hence, in spite of their apparent dissimilarity, to be essentially the same space, von Neumann named all spaces of this structure after Hilbert. The fact that this isomorphism entails the equivalence between Heisenberg's matrix mechanics and Schrödinger's wave mechanics made von Neumann aware of the importance of Hilbert spaces for the mathematical formulation of quantum mechanics.

To review this formulation let us recall some of its fundamental notions. A (closed) *subspace* S of a Hilbert space $\mathcal{H}$ is a *linear manifold* of vectors (i.e., closed under vector addition and multiplication by scalars) which is closed in the metric and hence a Hilbert space in its own right. The *orthogonal complement* $S^{\perp}$ of S is the set of all vectors which are orthogonal to all vectors of S. A mapping $\psi\rightarrow\varphi=A\psi$ of a linear manifold $\mathcal{D}_A$ into $\mathcal{H}$ is a *linear operator* A, with *domain* $\mathcal{D}_A$, if $A(a\psi_1+b\psi_2)=aA\psi_1+bA\psi_2$ for all ψ_1,ψ_2 of $\mathcal{D}_A$ and all a,b of $\mathcal{F}$. The image of $\mathcal{D}_A$ under A is the *range* $\mathcal{R}_A$ of A. The linear operator A is *continuous* if and only if it is *bounded* [i.e., if and only if $\|A\psi\|/\|\psi\|$ is bounded, where $\|\psi\|$ denotes the *norm* $(\psi,\psi)^{1/2}$ of ψ]. A' is an *extension* of A, or $A'\supseteq A$, if it coincides with A on $\mathcal{D}_A$ and $\mathcal{D}_{A'}\supseteq\mathcal{D}_A$. Since every bounded linear operator has a unique continuous extension to $\mathcal{H}$, its domain can always be taken as $\mathcal{H}$.

The *adjoint* A^{+} of a bounded linear operator A is the unique operator A^{+} which satisfies $(\varphi,A\psi)=(A^{+}\varphi,\psi)$ for all φ,ψ of $\mathcal{H}$. A is *self-adjoint* if $A=A^{+}$. A is unitary if $AA^{+}=A^{+}A=I$, where I is the identity operator. If S is a subspace of $\mathcal{H}$, then every vector ψ can uniquely be written $\psi=\psi_S^{+}\psi_{S^{\perp}}$, where ψ_S is in S and $\psi_{S^{\perp}}$ is in $S^{\perp}$, so that the mapping $\psi\rightarrow\psi_S=P_S\psi$ defines the *projection* P_S as a bounded self-adjoint idempotent (i.e., $P_S^2=P_S$) linear operator. Conversely, if a linear operator P is

Mathematics (Hawthorn, New York, 1970), pp. 550–576; M. Kline, *Mathematical Thought from Ancient to Modern Times* (Oxford University Press, New York, 1972), pp. 1091–1095.

bounded, self-adjoint, and idempotent, it is a projection. Projections and subspaces correspond one to one. The subspaces S and T are *orthogonal* [i.e., $(\varphi,\psi)=0$ for all φ of S and all ψ of T], in which case we also say that P_S and P_T are orthogonal if and only if $P_S P_T = P_T P_S = 0$ (null operator); and $\Sigma_{j=1}^N P_{S_j}$ is a projection if and only if $P_{S_j} P_{S_k} = 0$ for $j \neq k$.

$S \subset T$ (i.e., the subspace S is a subspace of T, in which case we also write $P_S < P_T$) if and only if $P_S P_T = P_T P_S = P_S$. In this case $P_T - P_S$ is a projection into the orthogonal complement of S in T, that is, the set of all vectors of T which are orthogonal to every vector of S.

For an unbounded linear operator A—which if it is symmetric [i.e., if $(\varphi,A\psi)=(A\varphi,\psi)$ for all φ,ψ of $\mathfrak{D}_A$] cannot, according to the Hellinger-Toeplitz theorem, have a domain which is $\mathcal{H}$ but may have a domain which is dense in $\mathcal{H}$—the self-adjoint is defined as follows. The set of all vectors φ for which there exists a vector φ^* such that $(\varphi,A\psi)=(\varphi^*,\psi)$ for all ψ of $\mathfrak{D}_A$ is the domain $\mathfrak{D}_{A^+}$ of the adjoint of A and the adjoint A^+ of A is defined by the mapping $\varphi \to \varphi^* = A^+\varphi$. A is self-adjoint if $A = A^+$.

According to the *spectral theorem*,[5] to every self-adjoint linear operator A corresponds a unique *resolution of identity*, that is, a set of projections $E^{(A)}(\lambda)$ or briefly E_λ, parametrized by real λ, such that (1) $E_\lambda < E_{\lambda'}$ for $\lambda < \lambda'$, (2) $E_{-\infty}=0$, (3) $E_\infty = I$, (4) $E_{\lambda+0}=E_\lambda$, (5) $I=\int_{-\infty}^{\infty} dE_\lambda$, (6) $A = \int_{-\infty}^{\infty}\lambda\, dE_\lambda$ [which is an abbreviation of $(\varphi,A\psi)=\int_{-\infty}^{\infty}\lambda\, d(\varphi,E_\lambda\psi)$, where the integral is to be interpreted as the Lebesgue-Stieltjes integral[6]], and finally (7) for all λ, E_λ commutes with any operator that commutes with A. The *spectrum* of A is the set of all λ which are not in an interval in which E_λ is constant. Those λ at which E_λ is discontinuous ("jumps") form the *point spectrum* which together with the *continuous spectrum* constitutes the *spectrum*.

Now, λ is an *eigenvalue* of A if there exists a nonzero vector φ called *eigenvector* belonging to λ, in $\mathfrak{D}_A$ such that $A\varphi=\lambda\varphi$. An eigenvalue is

[5]This theorem was proved by von Neumann in "Allgemeine Eigenwerttheorie Hermitischer Funktionaloperatoren," *Mathematische Annalen*, **102**, 49–131 (1929), reprinted in J. von Neumann, *Collected Works*, A. H. Taub, ed. (Pergamon Press, New York, 1961), Vol. 2, pp. 3–85. It was proved independently by M. H. Stone using a method earlier applied by T. Carleman to the theory of integral equations with singular kernel, cf. M. H. Stone, *Linear Transformations in Hilbert Space* (American Mathematical Society Colloquium Publications, Vol. 15, New York, 1932), Ch. 5. Other proofs were given by F. Riesz in 1930, B. O. Koopman and J. L. Doob in 1934, B. Lengyel in 1939, J. L. B. Cooper in 1945, and E. R. Lorch in 1950.

[6] $\int_a^b f(\lambda)dg(\lambda)$ is defined as $\lim \Sigma_{j=1}^n f(\lambda_j')[g(\lambda_{j+1})-g(\lambda_j)]$, where $\lambda_1,\lambda_2,\ldots,\lambda_n$ is a partition of the interval $[a,b]$, λ_j' is in the jth interval, and the limes denotes the passage to $\lambda_{j+1}-\lambda_j=0$ for all j.

nondegenerate if the subspace formed by the eigenvectors belonging to this eigenvalue is one-dimensional.[7] Every λ in the point spectrum of A is an eigenvalue of A. If the spectrum of A is a nondegenerate point spectrum $\lambda_j (j=1,2,\ldots)$, then the *spectral decomposition* (6) of A reduces to $A=\Sigma\lambda_j P_j$, where P_j is the projection on the eigenvector ("ray") φ_j belonging to λ_j. In fact, in this case $dE_\lambda = E_{\lambda+d\lambda} - E_\lambda \neq 0$ only if λ_j lies in $[\lambda, \lambda + d\lambda)$ where dE_λ becomes P_j. To vindicate this conclusion by an elementary consideration, let $\psi = \Sigma\varphi_j(\varphi_j, \psi)$ be an expansion of any vector ψ in terms of the eigenvectors φ_j of A; then $A\psi = \Sigma\lambda_j\varphi_j(\varphi_j,\psi) = \Sigma\lambda_j P_j\psi$ for all ψ.

With these mathematical preliminaries in mind and following von Neumann, we now give an axiomatized presentation of the formalism of quantum mechanics. The primitive (undefined) notions are *system*, *observable* (or "physical quantity" in the terminology of von Neumann), and *state*.

AXIOM I. To every system corresponds a Hilbert space $\mathcal{H}$ whose vectors (state vectors, wave functions) completely describe the states of the system.

AXIOM II. To every observable $\mathcal{A}$ corresponds uniquely a self-adjoint operator A acting in $\mathcal{H}$.

AXIOM III. For a system in state φ, the probability $\text{prob}_A(\lambda_1, \lambda_2 | \varphi)$ that the result of a measurement of the observable $\mathcal{A}$, represented by A, lies between λ_1 and λ_2 is given by $\|(E_{\lambda_2} - E_{\lambda_1})\varphi\|^2$, where E_λ is the resolution of the identity belonging to A.

AXIOM IV. The time development of the state vector φ is determined by the equation $H\varphi = i\hbar\partial\varphi/\partial t$ (Schrödinger equation), where the Hamiltonian H is the evolution operator and $\hbar$ is Planck's constant divided by 2π.

AXIOM V. If a measurement of the observable $\mathcal{A}$, represented by A, yields a result between λ_1 and λ_2, then the state of the system immediately after the measurement is an eigenfunction of $E_{\lambda_2} - E_{\lambda_1}$.

The *correspondence Axioms* I and II associate the primitive notions with mathematical entities. Von Neumann's original assumption that observables and self-adjoint operators stand in a one-to-one correspondence and that all nonzero vectors of the Hilbert space are state vectors had to be

[7]The dimension of a Hilbert space is the cardinality of a complete orthonormal system of vectors in it.

abandoned in view of the existence of superselection rules, discovered in 1952 by G. C. Wick, E. P. Wigner, and A. S. Wightman.

The often postulated statement that the result of measuring an observable $\mathcal{A}$, represented by A, is an element of the spectrum of A follows as a logical consequence from Axiom III. Moreover, the theorem that the expectation value $\mathrm{Exp}_\varphi A$ of $\mathcal{A}$ for a system in state φ, defined by the self-explanatory expression $\lim_{\Delta\to 0}\Sigma_j\lambda_j\ \mathrm{prob}_A(\lambda_j,\lambda_j+\Delta|\varphi)$, is $(\varphi,A\varphi)$ can easily be proved on the basis of Axioms I to III. Conversely, by the technique of characteristic functions as used in the theory of probability, it can be shown that this theorem entails Axiom III. Let us add that in the simple nondegenerate discrete case the just-mentioned definition of $\mathrm{Exp}_\varphi A$ becomes $\Sigma\lambda_j\mathrm{prob}_A(\lambda_j|\varphi)$, where, according to Axiom III, this probability $\mathrm{prob}_A(\lambda_j|\varphi)$ is given by $|(\varphi_j,\varphi)|^2$.

"Quantum statics," the part of quantum mechanics which disregards changes in time, is based, as we see, essentially only on one axiom, Axiom III. This axiom, moreover, is the only one which establishes some connection between the mathematics and physical data and therefore plays a major role for all questions of interpretations. In its ordinary interpretation it contains as a particular case Born's well-known probabilistic interpretation of the wave function according to which for a measurement of the position observable $\mathcal{Q}$ the probability density of finding the system at the position q is given by $|\psi(q)|^2$. In fact, if the operator Q, representing the observable $\mathcal{Q}$, is defined by $Q\psi(q)=q\psi(q)$, its spectral decomposition is given by $E_\lambda\psi(q)=\psi(q)$ for $q\leqslant\lambda$ and $E_\lambda\psi(q)=0$ for $q>\lambda$ and hence, according to Axiom III, the probability that $\lambda_1<q<\lambda_2$ is $\|(E_{\lambda_2}-E_{\lambda_1})\psi\|^2 = \int_1^2|\psi(q)|^2dq$, which proves the contention.

Axiom IV, the axiom of "quantum dynamics," can be replaced by postulating a one-parameter group of unitary operators $U(t)$ acting on the Hilbert space of the system such that $\varphi(t)=U(t)\varphi(0)$, and applying Stone's theorem according to which there exists a unique self-adjoint operator H such that $U(t)=\exp(-itH)$; it may also be equivalently formulated in terms of the statistical operator. Finally, Axiom V states that in the discrete case, immediately after having obtained the eigenvalue λ_j of A when measuring $\mathcal{A}$, the state of the system is an eigenvector of P_j, the projection on the eigenvector belonging to λ_j; for this reason Axiom V is called the "projection postulate." It is more controversial than the rest and has indeed been rejected by some theorists on grounds to be discussed in due course.

Although a complete derivation of all quantum mechanical theorems, with the inclusion of those pertaining to simultaneous measurements and identical particles, would require some additional postulates, these five axioms suffice for our purpose to characterize von Neumann's formalism of quantum mechanics, which is the one generally accepted.

In addition to the notions of system, observable, and state, the notions of *probability* and *measurement* have been used without interpretations. Although von Neumann used the concept of probability, in this context, in the sense of the frequency interpretation, other interpretations of quantum mechanical probability have been proposed from time to time. In fact, all major schools in the philosophy of probability, the subjectivists, the a priori objectivists, the empiricists or frequency theorists, the proponents of the inductive logic interpretation and those of the propensity interpretation, laid their claim on this notion. The different interpretations of probability in quantum mechanics may even be taken as a kind of criterion for the classification of the various interpretations of quantum mechanics. Since the adoption of such a systematic criterion would make it most difficult to present the development of the interpretations in their historical setting it will not be used as a guideline for our text.[8]

Similar considerations apply a fortiori to the notion of measurement in quantum mechanics. This notion, however it is interpreted, must somehow combine the primitive concepts of system, observable, and state and also, through Axiom III, the concept of probability. Thus measurement, the scientist's ultimate appeal to nature, becomes in quantum mechanics the most problematic and controversial notion because of its key position.

The major part of the operator calculus in Hilbert space and, in particular, its spectral theory had been worked out by von Neumann before Paul Adrien Maurice Dirac published in 1930 his famous treatise[9] in which he presented a conceptually most compact and notationally most elegant formalism for quantum mechanics. Even though von Neumann admitted that Dirac's formalism could "scarcely be surpassed in brevity and elegance," he criticized it as deficient in mathematical rigor, especially in view of its extensive use of the (at that time) mathematically unacceptable delta-function. Later, when Laurent Schwartz' theory of distributions made it possible to incorporate Dirac's improper functions into the realm of rigorous mathematics—a classic example of how physics may stimulate

[8]The reader interested in working out such a classification will find for his convenience bibliographical references in Selected Bibliography I in the Appendix at the end of this chapter. M. Strauss' essay "Logics for quantum mechanics," *Foundations of Physics* **3**, 265–276 (1973), contains useful suggestions of how to carry out such a classification.

[9]P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1930, 1935, 1947, 1958); *Die Prinzipien der Quantenmechanik* (Hirzel, Leipzig, 1930); *Les Principes de la Mécanique Quantique* (Presses Universitaires de France, Paris, 1931); *Osnovi Kvantovoj Mekhaniki* (GITTL, Moscow, Leningrad, 1932, 1937).

the growth of new branches in mathematics—Dirac's formalism seemed not to be assimilable to von Neumann's.[10] Yet due to its immediate intuitability and notational convenience Dirac's formalism not only survived but became the favorite framework for many expositions of the theory. The possibility of assimilating Dirac's formalism with von Neumann's approach has recently become the subject of important investigations such as Marlow's[11] presentation of the spectral theory in terms of direct integral decompositions of Hilbert space, Roberts'[12] recourse to "rigged" Hilbert spaces as well as the investigations by Hermann[13] and Antoine.[14]

Other formalisms of quantum mechanics such as the algebraic approach, initiated in the early 1930s by von Neumann, E. P. Wigner, and P. Jordan and elaborated in the 1940s by I. E. Segal, or the quantum logical approach, started by G. Birkhoff and von Neumann in 1936 and perfected by G. Mackey in the late 1950s, the former leading to the C^*-algebra theory of quantum mechanics and the latter to the development of modern quantum logic, will be discussed in their appropriate contexts. On the other hand, we shall hardly feel the need to refer to the S-matrix approach, which, anticipated in 1937 by J. A. Wheeler,[15] was developed in 1942 by Werner Heisenberg[16] for elementary particle theory—although it has recently been claimed[17] to be the most appropriate mathematical framework for a "pragmatic version" of the Copenhagen interpretation of the theory. Nor shall we have many occasions to refer to the interesting path integral

[10] Von Neumann apparently rejected this possibilty: "It should be emphasized that the correct structure does not consist in a mathematical refinement and explication of the Dirac method but rather necessitates a procedure differing from the very beginning, namely, the reliance on the Hilbert theory of operators." Preface, Ref. 2.

[11] A. R. Marlow, "Unified Dirac-von Neumann formulation of quantum mechanics," *Journal of Mathematical Physics* **6**, 919–927 (1965).

[12] J. E. Roberts, "The Dirac bra and ket formalism," *Journal of Mathematical Physics* **7**, 1097–1104 (1966); "Rigged Hilbert spaces in quantum mechanics," *Communications in Mathematical Physics*, **3**, 98–119 (1966).

[13] R. Hermann, "Analytic continuation of group representations," *Communications in Mathematical Physics* **5**, 157–190 (1967).

[14] J. P. Antoine, "Dirac formalism and symmetry problems in quantum mechanics," *Journal of Mathematical Physics* **10**, 53–69, 2276–2290 (1969).

[15] J. A. Wheeler, "On the mathematical description of light nuclei by the method of resonating group structure," *Physical Review* **52**, 1107–1122 (1937).

[16] W. Heisenberg, "'Beobachtbare Grössen' in der Theorie der Elementarteilchen," *Zeitschrift für Physik* **120**, 513–538 (1942).

[17] H. P. Stapp, "S-matrix interpretation of quantum mechanics," *Physical Review* **D3**, 1303–1320 (1971); "The Copenhagen interpretation," *American Journal of Physics* **40**, 1098–1116 (1972).