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TOPICS IN THE THEORY OF LIFTING

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Preface

The problem as to whether or not there exists a lifting of the M_R^∞ space[1]) corresponding to the real line and Lebesgue measure on it was first raised by A. Haar. It was solved in a paper published in 1931 [102] by J. von Neumann, who established the existence of a lifting in this case. In subsequent papers J. von Neumann and M. H. Stone [105], and later on J. Dieudonné [22], discussed various algebraic aspects and generalizations of the problem.

Attemps to solve the problem as to whether or not there exists a lifting for an arbitrary M_R^∞ space were unsuccessful for a long time, although the problem had significant connections with other branches of mathematics.

Finally, in a paper published in 1958 [88], D. Maharam established, by a delicate argument, that a lifting of M_R^∞ always exists (for an arbitrary space of σ-finite mass). D. Maharam proved first the existence of a lifting of the M_R^∞ space corresponding to a product $X=\prod_{i\in I}\{a_i,b_i\}$ and a product measure $\mu=\bigotimes_{i\in I}\mu_i$, with $\mu_i\{a_i\}=\mu_i\{b_i\}=\frac{1}{2}$ for all $i\in I$. Then, she reduced the general case to this one, via an isomorphism theorem concerning homogeneous measure algebras [87], [88].

A different and more direct proof of the existence of a lifting was subsequently given by the authors in [65]. A variant of this proof is presented in chapter 4.

It should be noticed that it is the "positivity" of the linear lifting (or correspondingly the "continuity property") which makes the proof of its existence difficult. However, it is precisely this property of the lifting which is important in applications.

The fact that there always exists a lifting (for strictly localizable spaces) has important applications. For instance: in the representation of integral operators, in the problem of disintegration of measures and ergodic theory, in the existence of separable modifications for stochastic processes and in the problem of point realizations of automorphisms

[1]) This notation is explained in section 7, chapter 1.

of L_R^∞ spaces[2]). Many classical theorems can be stated and proved in final form using the notion of lifting.

Most of the results given in this volume are due to the authors and were established in their papers quoted in the bibliography. The presentation makes use of various improvements in methods and results which were obtained subsequently by the authors.

Results concerning liftings commuting with sets of mappings (for instance the fact that if X is a locally compact group and μ a left invariant Haar measure on X, then there is always a lifting of the corresponding M_R^∞ space commuting with the left translation of X) are not included here. Complete details can be found in [54] and [72].

The volume is divided into ten main chapters. In chapter 1 we outline the setting for the theory of integration that is used in the book. The approach that we develop is based on the notion of upper integral. It provides a unified treatment for Bourbaki's integrals (both the usual and the essential integral) and for the integral in the setting of abstract measure spaces.

In chapter 2 we introduce the basic notion of admissible subalgebra of M_R^∞ and we study projections onto admissible subalgebras. Admissible subalgebras are used in the proof of the existence of a lifting.

In chapter 3 we introduce the notions of lifting, linear lifting and lower density (for an admissible subalgebra). We show in particular that the existence of a linear lifting is equivalent with the existence of a lifting.

The existence of a lifting (for strictly localizable spaces) is proved in chapter 4. The proof makes use of admissible subalgebras and an ergodic theorem concerning increasing sequences of projections (i. e. the appropriate version of the martingale convergence theorem, which is proved in Appendix I at the end of the volume). In view of later applications, we also show here how to define liftings for functions having as range a completely regular space.

In chapter 5 we define and study the topologies associated with lower densities and with liftings of an admissible subalgebra. Most of the results of this chapter were proved in [57].

In chapter 6 we discuss the integrability and measurability of functions with values in Banach spaces. The definitions and results are used in the next chapter.

In chapter 7 we prove a general integral representation theorem (without any separability assumptions). This theorem yields as corol-

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[2]) For further results concerning liftings and differentiation of measures, the reader may consult the paper [78], which appeared after the manuscript of this volume was completed and sent for publication.

laries the Dunford-Pettis theorem and the Dunford-Pettis-Phillips theorem. The dual of L_E^p $(1 \leqslant p < \infty)$ is obtained without any separability hypotheses. This chapter contains also a proof of Strassen's integral representation theorem, again without any separability assumptions. Using liftings for functions having as range a completely regular space, we give a short and direct proof of the fact that certain stochastic processes admit separable modifications.

In chapter 8 we introduce and discuss the notion of strong lifting and almost strong lifting, in the setting of locally compact spaces and Radon measures. It is shown that if X is metrizable and μ is a Radon measure on X having X for support, then there is a strong lifting of the corresponding M_R^∞ space. A series of examples in which X is not metrizable but a strong lifting still exists, are given (if X is a locally compact group and μ a left invariant Haar measure on X, then there exists a strong lifting). The chapter ends with a discussion of strong and almost strong liftings in the setting of topological (non locally compact) spaces. There is also an appendix on Borel liftings.

In chapter 9 we first prove a theorem about domination of measures, as a corollary to Strassen's theorem proved in chapter 7. From this theorem we obtain directly the disintegration of measures in the case of compact spaces and continuous mappings. The general disintegration theorem is proved in the last section of the chapter. We also show in this chapter that the existence of a strong lifting and that of a disintegration are in a certain sense equivalent problems. Among the notions introduced in this chapter in connection with the study of disintegration of measures we would like to mention that of "appropriate family".

In chapter 10 we show that every automorphism of an L_R^∞ space is induced by a point mapping.

The volume ends with two appendices and a short list of open problems (the main one, at present, being the existence of a strong lifting).

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Contents

CHAPTER I

Measure and integration

In this chapter we outline the setting for the theory of measure and integration that will be used in this book. The approach that we develop is essentially based on the notion of upper integral. It has the advantage that it provides a unified treatment for Bourbaki's integrals (both the usual and the "essential" integral) and for the integral in abstract measure spaces. This chapter is self-contained. We give complete definitions and complete statements of the most important results in the theory. The proofs however are omitted in most cases.

1. The upper integral

Let X be a set and let $\bar{R}_+^X$ be the set of all mappings $f: X \to \bar{R}_+$ (here $\bar{R}$ is the extended real line and $\bar{R}_+ = \{x \in \bar{R} | x \geqslant 0\} = [0, +\infty]$).

Definition 1. – *A mapping* $N: \bar{R}_+^X \to \bar{R}_+$ *is an upper integral on* X *if it satisfies the following conditions:*

a) $N(0) = 0$;

b) $N(\lambda f) = \lambda N(f)$ *for all* $f \in \bar{R}_+^X$ *and all* $\lambda > 0$;

c) *If* $f \leqslant \sum_{n=1}^{\infty} g_n$ *then* $N(f) \leqslant \sum_{n=1}^{\infty} N(g_n)$;

d) *If* (f_n) *is an increasing sequence then* $N(\sup_n f_n) = \sup_n N(f_n)$.

If N is an upper integral on X and if for each $1 \leqslant p < +\infty$ we define $N_p: \bar{R}_+^X \to \bar{R}_+$ by the equations

$$N_p(f) = (N(f^p))^{1/p}, \quad f \in \bar{R}_+^X$$

then N_p is an upper integral on X.

From this point, up to and including section 8, N will denote a fixed upper integral on X.

A function $f \in \bar{R}^X$ is called *N-negligible* if $N(|f|)=0$. A set $A \subset X$ is called *N-negligible* if φ_A is *N-negligible.*

When there is no ambiguity we shall say *negligible* instead of *N-negligible.*

We say that a property $P(x)$ holds *N-almost everywhere on a set* $A \subset X$ or equivalently holds *for almost every* $x \in A$ *with respect to N* if the set of all $x \in A$ for which $P(x)$ is not true is *N-negligible.*

When there is no ambiguity we shall say almost everywhere instead of N-almost everywhere and for almost every $x \in A$ instead of for almost every $x \in A$ with respect to N.

We shall now point out several properties of *N-negligible sets* and *N-negligible functions.*

(1) If A is negligible and $B \subset A$, then B is negligible.

(2) If (A_n) is a sequence of negligible sets then $\bigcup_n A_n$ and $\bigcap_n A_n$ are negligible.

(3) A function $f \in \bar{R}^X$ is negligible if and only if $S_f = \{x \mid f(x) \neq 0\}$ is negligible.

(4) If $f \in \bar{R}^X$ and $N(|f|) < +\infty$ then the set $\{x \mid |f(x)| = +\infty\}$ is negligible, i.e., f is finite almost everywhere.

Let $f: X \to \bar{R}$. For each $1 \leqslant p < +\infty$ we define $N_p(f)$ by

$$N_p(f) = N_p(|f|).$$[1]

We denote by $\mathscr{F}^p(X,N)$ the set of all $f: X \to R$ for which

$$N_p(f) < +\infty.$$

The restriction of N_p to $\mathscr{F}^p(X,N)$ will be denoted by the same symbol.

When there is no ambiguity we shall write $\mathscr{F}^p$ instead of $\mathscr{F}^p(X,N)$.

Theorem 1. – *For each $1 \leqslant p < +\infty$ the set $\mathscr{F}^p$ is a vector space, N_p is a seminorm on $\mathscr{F}^p$ and $\mathscr{F}^p$ is complete with respect to this seminorm.*

From now on we shall always assume that $\mathscr{F}^p$ is endowed with the topology defined by the seminorm N_p; we shall call this topology the *topology of mean convergence of order p* (for $p=1$ we call it simply the *topology of mean convergence*) *on* $\mathscr{F}^p$. If $A \subset \mathscr{F}^p$ then we shall usually call topology of mean convergence of order p on A (or topology of mean convergence if $p=1$) the topology induced on A by the topology of $\mathscr{F}^p$.

[1]) If $g: X \to \bar{R}$ and $h: X \to \bar{R}$ coincide almost everywhere then $N_p(f) = N_p(g)$ for every $1 \leqslant p < +\infty$.

2. The spaces $\mathscr{L}^p$ and L^p $(1 \leqslant p < +\infty)$

Let now $\mathscr{R} \subset \mathscr{F}^1(X,N)$ be a set with the following properties:

(L_1) *$\mathscr{R}$ is a vector space;*
(L_2) *If $\varphi \in C_R(R)$, $\varphi(0)=0$ and $f \in \mathscr{R}$, then $\varphi \circ f \in \mathscr{R}$;*
(L_3) *$N(f+g)=N(f)+N(g)$ for $f \in \mathscr{R}_+$ and $g \in \mathscr{R}_+$.*[1])

Remark. It follows easily from conditions (L_1) and (L_2) that $\mathscr{R}$ is a Riesz space (for the usual pointwise order relation) and that $\mathscr{R}_+$ spans $\mathscr{R}$. From (L_2) we also deduce:

(C_1) $f \in \mathscr{R} \Rightarrow \inf(f,1) \in \mathscr{R}$;
(C_2) $f \in \mathscr{R}$ and $1 \leqslant p < +\infty \Rightarrow |f|^p \in \mathscr{R}$;
(C_3) $\mathscr{R}$ is an algebra (note that for $f \in \mathscr{R}$ and $g \in \mathscr{R}$ we have $4fg = (f+g)^2 - (f-g)^2$ and use (C_2) and (L_1));
(C_4) $\mathscr{R} \subset \mathscr{F}^p(X,N)$ for each $1 \leqslant p < +\infty$.

Definition 2. *Let $1 \leqslant p < +\infty$. We define $\mathscr{L}^p(X,N,\mathscr{R})$ to be the closure of $\mathscr{R}$ in $\mathscr{F}^p(X,N)$. We define $L^p(X,N,\mathscr{R})$ to be the separated space associated with $\mathscr{L}^p(X,N,\mathscr{R})$.*

Clearly $\mathscr{L}^p(X,N,\mathscr{R})$ and $L^p(X,N,\mathscr{R})$ are vector spaces. The *canonical mapping* of $\mathscr{L}^p(X,N,\mathscr{R})$ onto $L^p(X,N,\mathscr{R})$ will be denoted by $f \to \tilde{f}$. Note that if f and g belong to $\mathscr{L}^p(X,N,\mathscr{R})$ then $\tilde{g} = \tilde{f}$ if and only if the set $\{x \mid g(x) \neq f(x)\}$ is N-negligible.

Recall that the norm on $L^p(X,N,\mathscr{R})$ (which will also be denoted by N_p) is defined by

$$N_p(\tilde{f}) = N_p(f)$$

for each $\tilde{f} \in L^p(X,N,\mathscr{R})$.

When there is no ambiguity we shall write $\mathscr{L}^p$ instead of $\mathscr{L}^p(X,N,\mathscr{R})$ and L^p instead of $L^p(X,N,\mathscr{R})$.

Remarks. – 1) We want to stress the fact that the definition of the space $\mathscr{L}^p(X,N,\mathscr{R})$ *depends essentially on the vector space $\mathscr{R}$*, while the definition of the spaces $\mathscr{F}^p(X,N)$ *depends only on the upper integral.*

2) A function $f: X \to R$ belongs to $\mathscr{L}^p(X,N,\mathscr{R})$ if and only if for each $\varepsilon > 0$ there is $f_\varepsilon \in \mathscr{R}$ such that $N_p(f - f_\varepsilon) < \varepsilon$.

3) The *completeness* of $\mathscr{L}^p(X,N,\mathscr{R})$ and $L^p(X,N,\mathscr{R})$ is a consequence of their definition and the completeness of $\mathscr{F}^p(X,N)$. Therefore $L^p(X,N,\mathscr{R})$ $(1 \leqslant p < +\infty)$ is a *Banach space* when endowed with the norm N_p.

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[1]) For any ordered vector space E and set $S \subset E$ we write $S_+ = \{x \in S \mid x \geqslant 0\}$.

4) If $f \in \mathscr{F}^p(X,N)$ coincides almost everywhere with a function belonging to $\mathscr{L}^p(X,N,\mathscr{R})$ then $f \in \mathscr{L}^p(X,N,\mathscr{R})$. In particular $f \in \mathscr{L}^p(X,N,\mathscr{R})$ if $f: X \to R$ is N-negligible.

Let $A \subset X$, $f: A \to \bar{R}$. We say that f is *defined almost everywhere* if CA is negligible. If $g: X \to \bar{R}$, we say that g is *equivalent* with f if $g(x) = f(x)$ almost everywhere.

A function f with values in $\bar{R}$, defined almost everywhere, is said to be *p-integrable* $(1 \leqslant p < +\infty)$ *with respect to* $(N,\mathscr{R})$ if it is equivalent with a function $g \in \mathscr{L}^p(X,N,\mathscr{R})$. If $p=1$ then instead of 1-integrable with respect to $(N,\mathscr{R})$ we shall usually say *integrable with respect to* $(N,\mathscr{R})$ or $(N,\mathscr{R})$*-integrable*.

A function f belongs therefore to $\mathscr{L}^p(X,N,\mathscr{R})$ $(1 \leqslant p < +\infty)$ if and only if it is p-integrable with respect to $(N,\mathscr{R})$, has for *domain* X and *takes values in* R.

If f is a function with values in $\bar{R}$, defined almost everywhere, we define $N_p(f) = N_p(g)$ $(1 \leqslant p < +\infty)$ if $g: X \to \bar{R}$ and g is equivalent with f. If f is p-integrable with respect to $(N,\mathscr{R})$ and if $g \in \mathscr{L}^p(X,N,\mathscr{R})$ is equivalent with f, then we shall sometimes write $\tilde{f} = \tilde{g}$. We define mean convergence of order p in the natural way for functions that are p-integrable with respect to $(N,\mathscr{R})$.

When there is no ambiguity we shall say p-integrable instead of p-integrable with respect to $(N,\mathscr{R})$ and integrable instead of $(N,\mathscr{R})$-integrable.

From this point, up to and including section 8, $\mathscr{R} \subset \mathscr{F}^1(X,N)$ *will be a fixed set with the properties* (L_1), (L_2), (L_3).

Several properties of the spaces $\mathscr{L}^p$ and $L^p (1 \leqslant p < +\infty)$ are given in the theorems below:

Theorem 2. – *Let $\mathscr{E} \subset \mathscr{L}^p$ be a set dense in $\mathscr{L}^p$. Then for every $f \in \mathscr{L}^p$ there exists a sequence (f_n) of functions belonging to $\mathscr{E}$ and having the following properties:*

2.1) *The sequence (f_n) converges to f in mean of order p;*

2.2) *The sequence $(f_n(x))$ converges to $f(x)$ almost everywhere;*

2.3) *There is $g: X \to \bar{R}_+$ with $N_p(g) < +\infty$ such that $|f_n| \leqslant g$ for each n.*

In particular the result in Theorem 2 is valid for $\mathscr{E} = \mathscr{R}$. Moreover if $\mathscr{E} = \mathscr{R}$ and if $|f(x)| \leqslant M$ for all $x \in X$, then we may suppose that $|f_n(x)| \leqslant M$ for all $x \in X$ and all n.

The next result shows, in particular, that $\mathscr{L}^p$ is a Riesz space.

Theorem 3. – *For each $f \in \mathscr{L}^p$, the function $|f|$ belongs to $\mathscr{L}^p$ and the mapping $f \to |f|$ of $\mathscr{L}^p$ into $\mathscr{L}^p$ is uniformly continuous. If $f \in \mathscr{L}^p$ and $g \in \mathscr{L}^p$ then* $\sup(f,g) \in \mathscr{L}^p$ *and* $\inf(f,g) \in \mathscr{L}^p$.

For $\tilde{f} \in L^p$ and $\tilde{g} \in L^p$ we write

$$\tilde{f} \leqslant \tilde{g} \Leftrightarrow f(x) \leqslant g(x) \text{ almost everywhere.}$$

It is clear that this is an order relation in L^p and that L^p is a Riesz space with respect to this order relation. Moreover the Riesz space L^p is completely reticulated as follows from:

Theorem 4. – *Let $H \subset L^p_+$ be a set directed for the relation $\leqslant$ and such that $\sup_{\tilde{h} \in H} N_p(\tilde{h}) < \infty$. Then H has a supremum $\tilde{h}_\infty$ in L^p and*

$$\lim_{\tilde{h} \in H} N_p(\tilde{h}_\infty - \tilde{h}) = 0.$$

For any increasing sequence $(\tilde{h}_n)$ of elements in H such that $\sup_n N_p(\tilde{h}_n) = \sup_{\tilde{h} \in H} N_p(\tilde{h})$ we have

$$\sup_n \tilde{h}_n = \tilde{h}_\infty \quad \text{and} \quad \lim_n N_p(\tilde{h}_\infty - \tilde{h}_n) = 0.$$

It is clear that there exists an increasing sequence $(\tilde{h}_n)$ of elements in H such that $\sup_n N_p(\tilde{h}_n) = \sup_{\tilde{h} \in H} N_p(\tilde{h})$.

Theorem 5 (Monotone convergence). – *Let (f_n) be an increasing sequence of functions belonging to $\mathscr{L}^p_+$; then the following two assertions are equivalent:*

i) *The pointwise supremum f of (f_n) is p-integrable;*
ii) $\sup_n N_p(f_n) < +\infty$.

If ii) *is satisfied then we have*

$$\lim_n N_p(f - f_n) = 0.$$

A very useful result is the following:

Theorem 6 (Lebesgue). – *Let (f_n) be a sequence of functions belonging to $\mathscr{L}^p$ with the following properties:*

1) *The sequence $(f_n(x))$ converges almost everywhere to a limit $f(x) \in R$;*

2) *There is $g: X \to \bar{R}_+$ with $N_p(g) < +\infty$ such that for each n, $|f_n(x)| \leqslant g(x)$ almost everywhere.*

Then the function f (defined almost everywhere) is p-integrable and the sequence (f_n) converges to f in mean of order p.

A useful characterization of $\mathscr{L}^p$ spaces is the following:

Theorem 7. – *A function $f: X \to R$ belongs to $\mathscr{L}^p$ if and only if $|f|^{p-1} f \in \mathscr{L}^1$.*

Let now $1 \leqslant p < +\infty$ and $A \subset X$. It follows from Theorem 7 that

$$\varphi_A \in \mathscr{L}^1 \Leftrightarrow \varphi_A \in \mathscr{L}^p.$$

We shall now close this section with the following result which is an immediate consequence of Theorems 2 and 6:

Theorem 8. – *Let $1 \leqslant p < +\infty$ and let $\varphi \in \mathscr{L}^p$ and $f \in \mathscr{L}^p$. Suppose φ bounded. Then $\varphi f \in \mathscr{L}^p$.*

From Theorem 8 it follows in particular that if $A \subset X$ is a set such that $\varphi_A \in \mathscr{L}^1$ and if $f \in \mathscr{L}^p$ then $\varphi_A f \in \mathscr{L}^p$.

3. The integral

In the previous section we defined the notion of integrable function. We shall now define the integral of an integrable function.

Definition 3. – *We call integral associated with $(N, \mathscr{R})$ the unique linear mapping $\mu_{(N,\mathscr{R})}: \mathscr{R} \to R$ satisfying*

$$\mu_{(N,\mathscr{R})}(f) = N(f) \quad \textit{for} \quad f \in \mathscr{R}_+.$$

When there is no ambiguity we write μ instead of $\mu_{(N,\mathscr{R})}$.

In other words, μ is the linear extension of $N|\mathscr{R}_+$ to the vector space $\mathscr{R}$ spanned by the cone $\mathscr{R}_+$ (see section 2, axiom (L_3) and the remark following it).

We remark that for each $f \in \mathscr{R}$ we have

$$|\mu(f)| \leqslant N(f).$$

Hence $\mu: \mathscr{R} \to R$ is a continuous linear mapping if $\mathscr{R}$ is endowed with the topology of mean convergence. It follows that there exists a *unique continuous linear extension* of μ to $\mathscr{L}^1$. The value of this linear functional for an element $f \in \mathscr{L}^1$ is denoted by $\int_X f\,d\mu$ (or $\int_X f\,d\mu_{(N,\mathscr{R})}$)[1]) or $\int f\,d\mu$ and is called *the integral of f*. Although the main properties of the integral are immediate consequences of the basic properties of the upper integral we shall point out the following:

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[1]) When necessary we write

$$\int_X f(x)\,d\mu(x) \quad \text{or} \quad \int_X f(x)\,d\mu_{(N,\mathscr{R})}(x).$$

(1) For $f \in \mathscr{L}^1_+$, $\int\limits_X f\,d\mu = N(f)$.

(2) For each $g \in \mathscr{L}^1$, $\left|\int\limits_X g\,d\mu\right| \leqslant \int\limits_X |g|\,d\mu$.

(3) If $f \in \mathscr{L}^1$, $g \in \mathscr{L}^1$ are equivalent, then $\int\limits_X f\,d\mu = \int\limits_X g\,d\mu$.

When $A \subset X$ is such that $\varphi_A \in \mathscr{L}^1$ we shall sometimes write $\mu(A)$ instead of $\int\limits_X \varphi_A\,d\mu$.

4. Measurable functions

For each $f \in \bar{R}^X_+$ we define

$$\mathscr{D}_f = \{g \text{ integrable} \mid g \geqslant f\}.$$

Definition 4. – *We say that the upper integral N is regular if, for every $f \in \bar{R}^X_+$,*

$$N(f) = \begin{cases} \inf\{N(g) \mid g \in \mathscr{D}_f\} & \text{if } \mathscr{D}_f \neq \varnothing, \\ +\infty \quad \text{if } \mathscr{D}_f = \varnothing. \end{cases}$$

From this point on we always assume that the upper integrals we consider are regular.

We shall now introduce two important definitions.

Definition 5. – *A set $A \subset X$ is $(N,\mathscr{R})$-integrable if $\varphi_A \in \mathscr{L}^1(X,N,\mathscr{R})$. We denote by $\mathscr{B}_0(X,N,\mathscr{R})$ the set of all $(N,\mathscr{R})$-integrable sets $A \subset X$.*

Remarks. – 1) If A and B belong to $\mathscr{B}_0(X,N,\mathscr{R})$ then $A \cup B$, $A \cap B$, and $A - B$ belong to $\mathscr{B}_0(X,N,\mathscr{R})$.

2) If $A \subset X$ is such that $N(\varphi_A) < \infty$ then there exists $B \in \mathscr{B}_0(X,N,\mathscr{R})$ such that $B \supset A$.

In fact, let $g \in \mathscr{D}_{\varphi_A}$; then $N(g) < \infty$. Let $g_1 = \inf(1,g)$. Then $\varphi_A \leqslant g_1^n \leqslant g_1$ and $g_1^n \in \mathscr{L}^1$ for all n (see Theorem 8). The sequence (g_1^n) converges pointwise to a characteristic function φ_B which (by Lebesgue's theorem) belongs to $\mathscr{L}^1$.

When there is no ambiguity we shall say integrable instead of $(N,\mathscr{R})$-integrable and we shall write $\mathscr{B}_0$ instead of $\mathscr{B}_0(X,N,\mathscr{R})$.

Definition 6. – *A function $f: X \to \bar{R}$ is called $(N,\mathscr{R})$-measurable if given any $B \in \mathscr{B}_0(X,N,\mathscr{R})$ there is a sequence (h_n) of functions belonging to $\mathscr{R}$ such that*

$$\lim_n h_n(x) = f(x)$$

N-almost everywhere on B. We denote by $\mathscr{L}(X,N,\mathscr{R})$ the set of all $(N,\mathscr{R})$-measurable functions on X to R.

From the definition it follows immediately that $\mathscr{L}(X,N,\mathscr{R})$ is an *algebra* over R and a *Riesz space* (for the usual pointwise order).

When there is no ambiguity we shall say measurable instead of $(N,\mathscr{R})$-measurable and we shall write $\mathscr{L}$ instead of $\mathscr{L}(X,N,\mathscr{R})$.

A set $A \subset X$ is said to be $(N,\mathscr{R})$-*measurable* if φ_A is $(N,\mathscr{R})$-measurable. We denote by $\mathscr{B}(X,N,\mathscr{R})$ the set of all $(N,\mathscr{R})$-measurable sets $A \subset X$.

Proposition 1. – *A set $A \subset X$ belongs to $\mathscr{B}(X,N,\mathscr{R})$ if and only if $A \cap B \in \mathscr{B}_0(X,N,\mathscr{R})$ for every $B \in \mathscr{B}_0(X,N,\mathscr{R})$.*

When there is no ambiguity we shall say (as in the case of functions) measurable instead of $(N,\mathscr{R})$-measurable and we shall write $\mathscr{B}$ instead of $\mathscr{B}(X,N,\mathscr{R})$.

Proposition 2. – *The set $\mathscr{B}$ is a tribe ($=\sigma$-algebra).*

Let $A \subset X$ be $(N,\mathscr{R})$-measurable and $f: A \to \bar{R}$. We say that f is $(N,\mathscr{R})$-measurable if the mapping $f': X \to \bar{R}$ defined by

$$f'(x) = \begin{cases} f(x) & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A \end{cases}$$

is $(N,\mathscr{R})$-measurable.

Remarks. – 1) Let f and g be two functions on X to $\bar{R}$ which coincide almost everywhere on each integrable set. If f is measurable then g is measurable.

2) Every p-integrable function $(1 \leqslant p < +\infty)$ is $(N,\mathscr{R})$-measurable (use Theorem 2).

3) If $f: X \to \bar{R}$ is measurable and $K \subset \bar{R}$ is a Borel set, then $f^{-1}(K)$ is measurable.

To prove 3) we reason as follows. We suppose first $f \in \mathscr{R}$. Assume that $K \subset \bar{R}$ is closed and such that $K \not\ni 0$. There is then a sequence (φ_n) of continuous functions on $\bar{R}$ to R which converges pointwise to φ_K; moreover we may suppose $\varphi_n(0)=0$ for all n. Then $(\varphi_n \circ f)$ converges pointwise to $\varphi_K \circ f = \varphi_{f^{-1}(K)}$. Since $\varphi_n \circ f \in \mathscr{R}$ for all n, we deduce that $\varphi_{f^{-1}(K)}$ is measurable. It follows then easily that $f^{-1}(K)$ is measurable for every Borel set $K \subset \bar{R}$.

Let now $f: X \to \bar{R}$ be measurable and let $K \subset \bar{R}$ be closed. We shall show that $f^{-1}(K)$ is measurable. For this it is enough to show that $B \cap f^{-1}(K)$ is measurable for every integrable set B. Let then $B \in \mathscr{B}_0(X,N,\mathscr{R})$ and let (h_n) be a sequence of functions in $\mathscr{R}$ which converges to $f(x)$ for every $x \in B_0$, where $B_0 \subset B$ and $B - B_0$ is negligible. If (U_n) is a sequence of open sets containing K and such that $\bigcap_n \bar{U}_n = K$, then we have

$$B_0 \cap \Big(\bigcap_{p \geqslant 1} \bigcup_{m \geqslant 1} \bigcap_{n \geqslant m} h_n^{-1}(U_p) \Big) = B_0 \cap f^{-1}(K).$$

We conclude that $B \cap f^{-1}(K)$ is measurable. Since B was arbitrary, $f^{-1}(K)$ is measurable. It follows then easily that $f^{-1}(K)$ is measurable for every Borel set $K \subset \bar{R}$.