ORDINARY DIFFERENTIAL EQUATIONS

SECOND EDITION

Garrett Birkhoff

Gian-Carlo Rota

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Preface

The theory of differential equations is distinguished for the wealth of its ideas and methods. Although this richness makes the subject attractive as a field of research, it has frequently been the cause of confusion on the part of the student. For many students the transition from the elementary theory of differential equations to the study of advanced methods and techniques has been too abrupt. One of the chief purposes of the present text is to fill this gap.

We present what seem to us to be the most important key ideas of the subject in their simplest context, often that of second-order equations. We have deliberately avoided the systematic elaboration of these key ideas, feeling that this is often best done by the student himself. After one has grasped the underlying methods, one of the best ways to develop technique is to generalize (say, to higher-order equations or systems) by one's own efforts.

The exposition presupposes primarily a knowledge of the advanced calculus and some experience with the formal manipulation of elementary differential equations. Beyond this, only an acquaintance with vectors, matrices, and elementary complex functions is assumed in most of the book. Familiarity with the concepts of pole and branch point is assumed in Chapter 9, and in Chapter 11 Euclidean vector spaces are used freely.

The book falls broadly into three parts. Chapters 1 through 4 constitute a review of material to which, presumably, the student has already been exposed in elementary courses. This has a twofold purpose: to fill the inevitable gaps in the student's knowledge of the beginnings of the subject and to give a rigorous presentation of the material. The first part covers elementary methods of integration of first-order, second-order linear, and nth-order linear constant-coefficient differential equations. Besides reviewing elementary methods, it includes rigorous discussions of comparison theorems and the method of majorants. Finally, a brief introduction is given to the concepts of transfer function and Nyquist diagram and their relation to the Laplace transform. Although widely used in electrical engineering for many years, these concepts seem not to have previously found their way into textbooks on differential equations.

Chapters 5 through 8 deal with systems of nonlinear differential equations. Chapter 5 treats plane autonomous systems, including the classification of nondegenerate critical points, and introduces the important notion of stability and Liapunov's method, which is then applied to some of the simpler types of nonlinear oscillations. Chapter 6 includes theorems of existence, uniqueness, and continuity, both in the small and in the large, and introduces the perturbation equations. Chapters 7 and 8 provide a brief survey of the theory of effective numerical integration.

Finally, Chapters 9 through 11 are devoted to the study of second-order linear differential equations. Chapter 9 develops the theory of regular singular points, with applications to some important special functions. Chapter 10 is devoted to Sturm-Liouville theory and related asymptotic formulas, for both finite and infinite intervals. Chapter 11 establishes the completeness of the eigenfunctions of regular Sturm-Liouville systems, without assuming the Lebesgue integral.

Throughout the book, the properties of various important special functions—notably Bessel functions, hypergeometric functions, and the more common orthogonal polynomials—are derived from their defining differential equations and boundary conditions. In this way we illustrate the theory of ordinary differential equations and show its power.

This textbook can be used either in a one-term survey course or as a leisurely one-year course. In a one-term course, the instructor would normally omit starred sections. Or he might try to cover thoroughly a selection of chapters, developing each in full, for example Chapters 1–3, 5 and 6, and 9–11—or, in a more elementary course, Chapters 1–8.

In a year's course, the book can be used as an introduction to more advanced and systematic treatments, such as those found in the well-known treatises of Cesari, Coddington and Levinson, Ince, and Nemytskii and Stepanoff. Another possibility is to use the book as a continuation of an elementary text on differential equations.

This text contains several hundred exercises of varying difficulty, which in all cases should be an important part of the course. The most difficult exercises are starred.

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Cambridge, Massachusetts

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I First-Order Differential Equations

1 Introduction

A *differential equation* is an equation between specified derivatives of a function, its values, and known quantities. Many laws of physics are most simply and naturally formulated as differential equations (or DE's, as we shall write for short). For this reason, DE's have been studied by the greatest mathematicians and mathematical physicists since the time of Newton.

Ordinary differential equations are DE's whose unknowns are functions of a single variable; they arise most commonly in the study of dynamic systems and electric networks. They are much easier to treat than partial differential equations, whose unknown functions depend on two or more independent variables.

Ordinary DE's are classified according to their order. The *order* of a DE is defined as the largest positive integer, n, for which an nth derivative occurs in the equation. This Chapter will be restricted to *real first-order* DE's of the form

$$\phi(x, y, y') = 0. \tag{1}$$

Given the function ϕ of three real variables, the problem is to determine all real functions $y = f(x)$ which satisfy the DE, that is, all solutions of (1) in the following sense.

■ DEFINITION. *A* solution *of* (1) *is a differentiable function* $f(x)$ *such that* $\phi(x, f(x), f'(x)) = 0$ *for all* x *in the interval where* $f(x)$ *is defined.*

EXAMPLE 1. In the first-order DE

$$x + yy' = 0, \tag{2}$$

the function ϕ is a polynomial function $\phi(x, y, z) = x + yz$ of the three variables involved. The solutions of (2) can be found by considering the identity $d(x^2 + y^2)/dx = 2(x + yy')$. From this identity, one sees that $x^2 + y^2 = C$ is a constant if $y = f(x)$ is any solution of (2).

The equation $x^2 + y^2 = C$ defines y *implicitly* as a two-valued function of x, for any positive constant C. Solving for y, we get *two* solutions, the (single-valued†) functions $y = \pm\sqrt{C - x^2}$, for each positive constant C. The *graphs* of these solutions, the so-called *solution curves*, form two families of semicircles, which fill the upper half-plane $y > 0$ and the lower half-plane $y < 0$, respectively.

On the x-axis, where $y = 0$, the DE (2) implies that $x = 0$. Hence the DE has no solutions which cross the x-axis, except possibly at the origin. This fact is easily overlooked, because the solution curves *appear* to cross the x-axis to form full circles, as in Figure 1.1. However, these

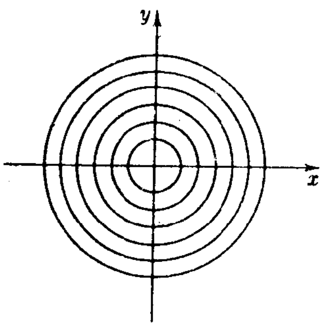

FIGURE 1.1 Integral Curves of $x + yy' = 0$.

circles have infinite slope where they cross the x-axis; hence y' does not exist, and the DE (2) is *not* satisfied there.

The preceding difficulty also arises if one tries to solve the DE (2) for y'. Dividing through by y, one gets $y' = -x/y$, an equation which cannot be satisfied if $y = 0$. The preceding difficulty is thus avoided if one restricts attention to regions where the DE (1) is normal, in the following sense.

■ DEFINITION. *A* normal *first-order* DE *is one of the form*

$$y' = F(x, y) \tag{3}$$

† In this book, the word "function" will always mean single-valued function, unless the contrary is expressly specified.

In the normal form $y' = -x/y$ of the DE (2), the function $F(x, y)$ is continuous in the upper half-plane $y > 0$ and in the lower half-plane where $y < 0$; it is undefined on the x-axis.

2 Fundamental theorem of the calculus

The most familiar class of differential equations consists of the first-order DE's of the form

$$y' = g(x). \tag{4}$$

Such DE's are normal; their solutions are described by the fundamental theorem of the calculus, which reads as follows.

■ FUNDAMENTAL THEOREM OF THE CALCULUS. *Let the function $g(x)$ in* DE (4) *be continuous in the interval $a \leqq x \leqq b$. Given a number c, there is one and only one solution $f(x)$ of the* DE (4) *in the interval such that $f(a) = c$. This solution is given by the definite integral*

$$f(x) = c + \int_a^x g(t)\,dt, \qquad c = f(a). \tag{5}$$

This basic result serves as a model of rigorous formulation in several respects. First, it specifies the region under consideration, as a vertical strip $a \leqq x \leqq b$ in the xy-plane. Second, it describes in precise terms the class of functions $g(x)$ considered. And third, it asserts the *existence* and *uniqueness* of a solution, given the "initial condition" $f(a) = c$.

We recall that the definite integral

$$\int_a^x g(t)\,dt = \lim_{\max \Delta t_k \to 0} \sum g(t_k)\,\Delta t_k, \qquad \Delta t_k = t_k - t_{k-1}, \tag{5'}$$

is defined for each fixed x as a limit of Riemann sums; it is not necessary to find a formal expression for the indefinite integral $\int g(x)\,dx$ to give meaning to the definite integral $\int_a^x g(t)\,dt$, provided only that $g(t)$ is continuous. Such functions as the *error function* $\operatorname{erf} x = (2/\sqrt{\pi}) \int_0^x e^{-t^2}\,dt$ and the *sine integral function* $\mathrm{SI}(x) = \int_x^\infty [(\sin t)/t]\,dt$ are indeed commonly *defined* as definite integrals; cf. Chapter 3, §1.

To formulate and prove analogous theorems for more general first-order normal DE's, we need some technical concepts. We define a *domain*† as a nonempty *open connected* set. A function $\phi = \phi(x_1, \ldots, x_r)$ is said to be *of class* $\mathscr{C}^n$ in a domain D, when all its derivatives $\partial\phi/\partial x_i$, $\partial^2\phi/\partial x_i\,\partial x_j$, $\cdots$ of orders $1, \cdots, n$ exist and are continuous in D. One writes this condition in symbols as $\phi \in \mathscr{C}^n$ in D, or $\phi \in \mathscr{C}^n(D)$. When ϕ

† Some authors say *region* where we say domain. We will call the closure of a domain a *closed domain*.

is merely assumed to be continuous in D, one writes $\phi \in \mathscr{C}$ in D, or $\phi \in \mathscr{C}(D)$.

Intervals appear so frequently in analysis that they are referred to by a special notation. Thus, the *closed* interval $a \leqq x \leqq b$, which is not a domain (why not?) is denoted by $[a, b]$, the *open* interval $a < x < b$ by (a, b), the positive semi-axis $0 \leqq x < +\infty$ by $[0, +\infty)$, and so on. Generally, a round bracket indicates that the endpoint adjacent to it is excluded from the interval, and a square bracket that the adjacent endpoint is included.

Given $F(x)$, the notation $F \in \mathscr{C}^2[1, +\infty)$ thus means that F is twice continuously differentiable in the semi-infinite line $[1, +\infty)$. Considered as a function of the two variables x and y, F is of class $\mathscr{C}^2$ in the *closed* domain including the vertical line $x = 1$ and all points to the right of it in the xy-plane. Likewise, $F \in \mathscr{C}[0, 1]$ means that F is continuous in the vertical strip $0 \leqq x \leqq 1$. Where there is any question of just what domain is referred to below, the domain will be described in words as well as in symbols.

There are a number of obvious facts about the differentiability of solutions of DE's. Such facts about differentiability will be used without special comment where they are irrelevant to the main idea of a proof. For instance, if $g \in \mathscr{C}^n(a, b)$, and $y = f(x)$ is any solution of the DE $y' = g(x)$, then $y \in \mathscr{C}^{n+1}(a, b)$. Again, if $\phi \in \mathscr{C}^n$ and $\psi \in \mathscr{C}^n$ in a domain D, and $F(u, v) \in \mathscr{C}^n$ in the entire uv-plane, then $G(x, y) = F(\phi(x, y), \psi(x, y)) \in \mathscr{C}^n(D)$.

3 Solutions and integrals

According to the definition given in §1, a *solution* of a DE is always a *function*. For example, the solutions of the DE $x + yy' = 0$ in Example 1 are the functions $y = \pm\sqrt{C - x^2}$, whose graphs are semicircles of arbitrary diameter, centered at the origin. The graph of the solution curves are, however, more easily described by the equation $x^2 + y^2 = C$, describing a family of circles centered at the origin. In what sense can such a family of curves be considered as a solution of the DE? To answer this question, we require a new notion.

■ DEFINITION. *An* integral *of* DE (1) *is a function of two variables, $u(x, y)$, which assumes a constant value whenever the variable y is replaced by a solution $y = f(x)$ of the* DE.

In the above example, the function $u(x, y) = x^2 + y^2$ is an integral of the DE $x + yy' = 0$, because, upon replacing the variable y by any function $\pm\sqrt{C - x^2}$, we obtain $u(x, y) = C$.

The second-order DE,

$$\frac{d^2x}{dt^2} = -x, \tag{2'}$$

becomes a first-order DE equivalent to (2) after setting $dx/dt = y$:

$$y\frac{dy}{dx} = -x. \tag{2}$$

As we have seen, the curves $u(x, y) = x^2 + y^2$ are integrals of this DE. When the DE (2′) is interpreted as an equation of motion under Newton's second law (see, for example, the discussion in Chapter 5, §7), the integrals $C = x^2 + y^2$ represent curves of constant energy C. This illustrates an important principle: an integral of a DE representing some kind of motion is a quantity that remains unchanged through the motion.

The relationship between *solutions* and *integrals* of the DE (1) will be made clear by the following theorem.

■ IMPLICIT FUNCTION THEOREM.† *Let $u(x, y)$ be a function of class $\mathscr{C}^n$ in a domain containing the point (x_0, y_0), and let $\partial u(x_0, y_0)/\partial y \neq 0$. Then there exists a unique function $y = f(x, C)$ of class $\mathscr{C}^n$, defined in some open interval (a, b) containing x_0, such that $y_0 = f(x_0, C)$ and $u(x, f(x, C)) = C$ for all x in (a, b) and for all C in an open interval.*

By the Implicit Function Theorem, every integral $u(x, y)$ of class $\mathscr{C}^1$ of the DE (1) defines a family of solutions near any point (x, y) where $\partial u(x, y)/\partial x \neq 0$, obtained by solving the equation $u(x, y) = C$ for the variable y.

The notion of integral has been defined in terms of a solution of a DE. For several classes of DE's, however, it is possible to verify that a function $u(x, y)$ is an integral without first finding any solution. For example, a function $u(x, y)$ of class $\mathscr{C}^1$ is an integral of the *quasilinear* DE

$$\phi(x, y, y') = M(x, y) + N(x, y)y' = 0$$

whenever

$$M(x, y)\frac{\partial u}{\partial y} - N(x, y)\frac{\partial u}{\partial x} = 0,$$

provided that $\partial u/\partial y \neq 0$, as can be verified from the familiar formula $dy/dx = -(\partial u/\partial x)/(\partial u/\partial y)$.

† Courant, Vol. 2, p. 114; Widder, p. 55. Here and below, page references to authors refer to the books listed in the selected bibliography on pp. 355–357.