Partial Differential Equations

Lawrence C. Evans

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ABSTRACT. This text surveys a wide variety of topics in the mathematical theory of partial differential equations (PDE). The primary topics are: representation formulas for solutions, theory for linear PDE, theory for nonlinear PDE.

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I dedicate this book to the memory of my parents,

LAWRENCE S. EVANS and LOUISE J. EVANS.

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PREFACE

I present in this book a wide-ranging survey of many important topics in the theory of partial differential equations (PDE), with particular emphasis on various modern approaches. I have made a huge number of editorial decisions about what to keep and what to toss out, and can only claim that this selection seems to me about right. I of course include the usual formulas for solutions of the usual linear PDE, but also devote large amounts of exposition to energy methods within Sobolev spaces, to the calculus of variations, to conservation laws, etc.

My general working principles in the writing have been these:

a. PDE theory is (mostly) not restricted to two independent variables. Many texts describe PDE as if functions of the two variables (x, y) or (x, t) were all that matter. This emphasis seems to me misleading, as modern discoveries concerning many types of equations, both linear and nonlinear, have allowed for the rigorous treatment of these in any number of dimensions. I also find it unsatisfactory to "classify" partial differential equations: this is possible in two variables, but creates the false impression that there is some kind of general and useful classification scheme available in general.

b. Many interesting equations are nonlinear. My view is that overall we know too much about linear PDE and too little about nonlinear PDE. I have accordingly introduced nonlinear concepts early in the text and have tried hard to emphasize everywhere nonlinear analogues of the linear theory.

c. Understanding generalized solutions is fundamental. Many of the partial differential equations we study, especially nonlinear first-order equations, do not in general possess smooth solutions. It is therefore essential to devise some kind of proper notion of generalized or weak solution. This is an important but subtle undertaking, and much of the hardest material in this book concerns the uniqueness of appropriately defined weak solutions.

d. PDE theory is not a branch of functional analysis. Whereas certain classes of equations can profitably be viewed as generating abstract operators between Banach spaces, the insistence on an overly abstract viewpoint, and consequent ignoring of deep calculus and measure theoretic estimates, is ultimately limiting.

e. Notation is a nightmare. I have really tried to introduce consistent notation, which works for all the important classes of equations studied. This attempt is sometimes at variance with notational conventions within a subarea.

f. Good theory is (almost) as useful as exact formulas. I incorporate this principle into the overall organization of the text, which is subdivided into three parts, roughly mimicking the historical development of PDE theory itself. Part I concerns the search for explicit formulas for solutions, and Part II the abandoning of this quest in favor of general theory asserting the existence and other properties of solutions for linear equations. Part III is the mostly modern endeavor of fashioning general theory for important classes of nonlinear PDE.

Let me also explicitly comment here that I intend the development within each section to be rigorous and complete (exceptions being the frankly heuristic treatment of asymptotics in §4.5 and an occasional reference to a research paper). This means that even locally within each chapter the topics do not necessarily progress logically from "easy" to "hard" concepts. There are many difficult proofs and computations early on, but as compensation many easier ideas later. The student should certainly omit on first reading some of the more arcane proofs.

I wish next to emphasize that this is a *textbook*, and not a reference book. I have tried everywhere to present the essential ideas in the clearest possible settings, and therefore have almost never established sharp versions of any of the theorems. Research articles and advanced monographs, many of them listed in the Bibliography, provide such precision and generality. My goal has rather been to explain, as best I can, the many fundamental ideas of the subject within fairly simple contexts.

I have greatly profited from the comments and thoughtful suggestions of many of my colleagues, friends and students, in particular: S. Antman, J. Bang, X. Chen, A. Chorin, M. Christ, J. Cima, P. Colella, J. Cooper,

M. Crandall, B. Driver, M. Feldman, M. Fitzpatrick, R. Gariepy, J. Goldstein, D. Gomes, O. Hald, W. Han, W. Hrusa, T. Ilmanen, I. Ishii, I. Israel, R. Jerrard, C. Jones, B. Kawohl, S. Koike, J. Lewis, T.-P. Liu, H. Lopes, J. McLaughlin, K. Miller, J. Morford, J. Neu, M. Portilheiro, J. Ralston, F. Rezakhanlou, W. Schlag, D. Serre, P. Souganidis, J. Strain, W. Strauss, M. Struwe, R. Temam, B. Tvedt, J.-L. Vazquez, M. Weinstein, P. Wolfe, and Y. Zheng.

I especially thank Tai-Ping Liu for many years ago writing out for me the first draft of what is now Chapter 11.

I am extremely grateful for the suggestions and lists of mistakes from earlier drafts of this book sent to me by many readers, and I encourage others to send me their comments, at evans@math.berkeley.edu. I have come to realize that I must be more than slightly mad to try to write a book of this length and complexity, but I am not yet crazy enough to think that I have made no mistakes. **I will therefore maintain a listing of errors which come to light, and will make this accessible through the math.berkeley.edu homepage.**

Faye Yeager at UC Berkeley has done a really magnificent job typing and updating these notes, and Jaya Nagendra heroically typed an earlier version at the University of Maryland. My deepest thanks to both.

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LCE

August, 1997

Berkeley

Chapter 1

INTRODUCTION

This chapter surveys the principal theoretical issues concerning the solving of partial differential equations.

To follow the subsequent discussion, the reader should first of all turn to Appendix A and look over the notation presented there, particularly the multiindex notation for partial derivatives.

1.1. PARTIAL DIFFERENTIAL EQUATIONS

A *partial differential equation (PDE)* is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Using the notation explained in Appendix A, we can write out symbolically a typical PDE, as follows. Fix an integer $k \geq 1$ and let U denote an open subset of $\mathbb{R}^n$.

DEFINITION. *An expression of the form*

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0 \quad (x \in U) \tag{1}$$

is called a k$^{\text{th}}$-order partial differential equation, *where*

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

is given, and

$$u : U \to \mathbb{R}$$

is the unknown.

We *solve* the PDE if we find all u verifying (1), possibly only among those functions satisfying certain auxiliary boundary conditions on some part Γ of ∂U. By finding the solutions we mean, ideally, obtaining simple, explicit solutions, or, failing that, deducing the existence and other properties of solutions.

DEFINITIONS.

(i) *The partial differential equation* (1) *is called* linear *if it has the form*

$$\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x)$$

for given functions a_α $(|\alpha| \le k), f$. *This linear PDE is* homogeneous *if* $f \equiv 0$.

(ii) *The PDE* (1) *is* semilinear *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0.$$

(iii) *The PDE* (1) *is* quasilinear *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0.$$

(iv) *The PDE* (1) *is* fully nonlinear *if it depends nonlinearly upon the highest order derivatives.*

A *system* of partial differential equations is, informally speaking, a collection of several PDE for several unknown functions.

DEFINITION. *An expression of the form*

$$\mathbf{F}(D^k\mathbf{u}(x), D^{k-1}\mathbf{u}(x), \dots, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0} \quad (x \in U) \tag{2}$$

is called a k^{th}-order system of partial differential equations, *where*

$$\mathbf{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$$

is given and

$$\mathbf{u} : U \to \mathbb{R}^m, \ \mathbf{u} = (u^1, \dots, u^m)$$

is the unknown.

Here we are supposing that the system comprises the same number m of scalar equations as unknowns $(u^1, \dots, u^m)$. This is the most common circumstance, although other systems may have fewer or more equations than unknowns.

Systems are classified in the obvious way as being linear, semilinear, etc.

Remark. We use "PDE" as an abbreviation for both "partial differential equation" and "partial differential equation*s*". □

1.2. EXAMPLES

There is no general theory known concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations that are important for applications within and outside of mathematics, with the hope that insight from the origins of these PDE can give clues as to their solutions.

Following is a list of many specific partial differential equations of interest in current research. This listing is intended merely to familiarize the reader with the names and forms of various famous PDE. To display most clearly the mathematical structure of these equations, we have mostly set relevant physical constants to unity. We will later discuss the origin and interpretation of many of these PDE.

Throughout $x \in U$, where U is an open subset of $\mathbb{R}^n$, and $t \ge 0$. Also $Du = D_x u = (u_{x_1}, \dots, u_{x_n})$ denotes the gradient of u with respect to the spatial variable $x = (x_1, \dots, x_n)$.

1.2.1. Single partial differential equations.

a. Linear equations.

1. *Laplace's equation*

$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0.$$

2. *Helmholtz's (or eigenvalue) equation*

$$-\Delta u = \lambda u.$$

3. *Linear transport equation*

$$u_t + \sum_{i=1}^n b^i u_{x_i} = 0.$$

4. *Liouville's equation*

$$u_t - \sum_{i=1}^n (b^i u)_{x_i} = 0.$$

5. *Heat (or diffusion) equation*

$$u_t - \Delta u = 0.$$

6. *Schrödinger's equation*

$$iu_t + \Delta u = 0.$$

7. *Kolmogorov's equation*

$$u_t - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

8. *Fokker–Planck equation*

$$u_t - \sum_{i,j=1}^{n} (a^{ij} u)_{x_i x_j} - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$

9. *Wave equation*

$$u_{tt} - \Delta u = 0.$$

10. *Telegraph equation*

$$u_{tt} + du_t - u_{xx} = 0.$$

11. *General wave equation*

$$u_{tt} - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

12. *Airy's equation*

$$u_t + u_{xxx} = 0.$$

13. *Beam equation*

$$u_t + u_{xxxx} = 0.$$

b. Nonlinear equations.

1. *Eikonal equation*

$$|Du| = 1.$$

2. *Nonlinear Poisson equation*

$$-\Delta u = f(u).$$

3. *p-Laplacian equation*

$$\operatorname{div}(|Du|^{p-2} Du) = 0.$$

4. *Minimal surface equation*

$$\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right) = 0.$$

5. *Monge–Ampère equation*

$$\det(D^2 u) = f.$$

6. *Hamilton–Jacobi equation*

$$u_t + H(Du, x) = 0.$$

7. *Scalar conservation law*

$$u_t + \operatorname{div} \mathbf{F}(u) = 0.$$

8. *Inviscid Burgers' equation*

$$u_t + uu_x = 0.$$

9. *Scalar reaction-diffusion equation*

$$u_t - \Delta u = f(u).$$

10. *Porous medium equation*

$$u_t - \Delta(u^\gamma) = 0.$$

11. *Nonlinear wave equations*

$$u_{tt} - \Delta u = f(u),$$
$$u_{tt} - \operatorname{div} \mathbf{a}(Du) = 0.$$

12. *Korteweg–de Vries (KdV) equation*

$$u_t + uu_x + u_{xxx} = 0.$$

1.2.2. Systems of partial differential equations.

a. Linear systems.

1. *Equilibrium equations of linear elasticity*

$$\mu\Delta\mathbf{u} + (\lambda+\mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

2. *Evolution equations of linear elasticity*

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u} - (\lambda+\mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

3. *Maxwell's equations*

$$\begin{cases} \mathbf{E}_t = \operatorname{curl}\mathbf{B} \\ \mathbf{B}_t = -\operatorname{curl}\mathbf{E} \\ \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0. \end{cases}$$

b. Nonlinear systems.

1. *System of conservation laws*

$$\mathbf{u}_t + \operatorname{div}\mathbf{F}(\mathbf{u}) = \mathbf{0}.$$

2. *Reaction-diffusion system*

$$\mathbf{u}_t - \Delta\mathbf{u} = \mathbf{f}(\mathbf{u}).$$

3. *Euler's equations for incompressible, inviscid flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

4. *Navier–Stokes equations for incompressible, viscous flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} - \Delta\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

See Zwillinger [**ZW**] for a much more extensive listing of interesting PDE.

1.3. STRATEGIES FOR STUDYING PDE

As explained in §1.1 our goal is the discovery of ways to solve partial differential equations of various sorts, but—as should now be clear in view of the many diverse examples set forth in §1.2—this is no easy task. And indeed the very question of what it means to "solve" a given PDE can be subtle, depending in large part on the particular structure of the problem at hand.

1.3.1. Well-posed problems, classical solutions.

The informal notion of a well-posed problem captures many of the desirable features of what it means to solve a PDE. We say that a given problem for a partial differential equation is *well-posed* if

(a) the problem in fact has a solution;

(b) this solution is unique;

and

(c) the solution depends continuously on the data given in the problem.

The last condition is particularly important for problems arising from physical applications: we would prefer that our (unique) solution changes only a little when the conditions specifying the problem change a little. (For many problems, on the other hand, uniqueness is not to be expected. In these cases the primary mathematical tasks are to classify and characterize the solutions.)

Now clearly it would be desirable to "solve" PDE in such a way that (a)–(c) hold. But notice that we still have not carefully defined what we mean by a "solution". Should we ask, for example, that a "solution" u must be real analytic or at least infinitely differentiable? This might be desirable, but perhaps we are asking too much. Maybe it would be wiser to require a solution of a PDE of order k to be at least k times continuously differentiable. Then at least all the derivatives which appear in the statement of the PDE will exist and be continuous, although maybe certain higher derivatives will not exist. Let us informally call a solution with this much smoothness a *classical* solution of the PDE: this is certainly the most obvious notion of solution.

So by solving a partial differential equation in the classical sense we mean if possible to write down a formula for a classical solution satisfying (a)–(c) above, or at least to show such a solution exists, and to deduce various of its properties.

1.3.2. Weak solutions and regularity.

But can we achieve this? The answer is that certain specific partial differential equations (e.g. Laplace's equation) can be solved in the classical sense, but many others, if not most others, cannot. Consider for instance the scalar conservation law

$$u_t + F(u)_x = 0.$$

We will see in §3.4 that this PDE governs various one-dimensional phenomena involving fluid dynamics, and in particular models the formation and propagation of shock waves. Now a shock wave is a curve of discontinuity of the solution u; and so if we wish to study conservation laws, and recover the underlying physics, we must surely allow for solutions u which are not continuously differentiable or even continuous. In general, as we shall see, the conservation law has no classical solutions, but *is* well-posed if we allow for properly defined *generalized* or *weak solutions*.

This is all to say that we may be forced by the structure of the particular equation to abandon the search for smooth, classical solutions. We must instead, while still hoping to achieve the well-posedness conditions (a)–(c), investigate a wider class of candidates for solutions. And in fact, even for those PDE which turn out to be classically solvable, it is often most expedient initially to search for some appropriate kind of weak solution.

The point is this: if from the outset we demand that our solutions be very regular, say k-times continuously differentiable, then we are usually going to have a really hard time finding them, as our proofs must then necessarily include possibly intricate demonstrations that the functions we are building are in fact smooth enough. A far more reasonable strategy is to consider as separate the *existence* and the *smoothness* (or *regularity*) problems. The idea is to define for a given PDE a reasonably wide notion of a *weak solution*, with the expectation that since we are not asking too much by way of smoothness of this weak solution, it may be easier to establish its existence, uniqueness, and continuous dependence on the given data. Thus, to repeat, it is often wise to aim at proving well-posedness in some appropriate class of weak or generalized solutions.

Now, as noted above, for various partial differential equations this is the best that can be done. For other equations we can hope that our weak solution may turn out after all to be smooth enough to qualify as a classical solution. This leads to the question of *regularity* of weak solutions. As we will see, it is often the case that the existence of weak solutions depends upon rather simple estimates plus ideas of functional analysis, whereas the regularity of the weak solutions, when true, usually rests upon many intricate calculus estimates.

Let me explicitly note here that once we are past Part I (Chapters 2–4), our efforts will be largely devoted to proving mathematically the existence

of solutions to various sorts of partial differential equations, and not so much to deriving formulas for these solutions. This may seem wasted or misguided effort, but in fact mathematicians are like theologians: we regard existence as the prime attribute of what we study. But unlike most theologians, we need not always rely upon faith alone.

1.3.3. Typical difficulties.

Following are some vague but general principles, which may be useful to keep in mind:

(1) Nonlinear equations are more difficult than linear equations; and, indeed, the more the nonlinearity affects the higher derivatives, the more difficult the PDE is.
(2) Higher-order PDE are more difficult than lower-order PDE.
(3) Systems are harder than single equations.
(4) Partial differential equations entailing many independent variables are harder than PDE entailing few independent variables.
(5) For most partial differential equations it is not possible to write out explicit formulas for solutions.

None of these assertions is without important exceptions.

1.4. OVERVIEW

This textbook is divided into three major Parts.

PART I: Representation Formulas for Solutions

Here we identify those important partial differential equations for which in certain circumstances explicit or more-or-less explicit formulas can be had for solutions. The general progression of the exposition is from direct formulas for certain linear equations, to far less concrete representation formulas, of a sort, for various nonlinear PDE.

Chapter 2 is a detailed study of four exactly solvable partial differential equations: the linear transport equation, Laplace's equation, the heat equation, and the wave equation. These PDE, which serve as archetypes for the more complicated equations introduced later, admit directly computable solutions, at least in the case that there is no domain whose boundary geometry complicates matters. The explicit formulas are augmented by various indirect, but easy and attractive, "energy"-type arguments, which serve as motivation for the developments in Chapters 6, 7 and thereafter.

Chapter 3 continues the theme of searching for explicit formulas, now for general first-order nonlinear PDE. The key insight is that such PDE can, locally at least, be transformed into systems of ordinary differential equations (ODE), the characteristic equations. We stipulate that once the problem becomes "only" the question of integrating a system of ODE, it is in principle solved, sometimes quite explicitly. The derivation of the characteristic equations given in the text is very simple and does not require any geometric insights. It is in truth so easy to derive the characteristic equations that no real purpose is had by dealing with the quasilinear case first.

We introduce also the Hopf–Lax formula for Hamilton–Jacobi equations (§3.3) and the Lax–Oleinik formula for scalar conservation laws (§3.4). (Some knowledge of measure theory is useful here, but is not essential.) These sections provide an early acquaintance with the global theory of these important nonlinear PDE, and so motivate the later Chapters 10 and 11.

Chapter 4 is a grab bag of techniques for explicitly (or kind of explicitly) solving various linear and nonlinear partial differential equations, and the reader should study only whatever seems interesting. The section on the Fourier transform is, however, essential. The Cauchy-Kovalevskaya Theorem appears at the very end. Although this is basically the only general existence theorem in the subject, and thus logically should perhaps be regarded as central, in practice these power series methods are not so prevalent.

PART II: Theory for Linear Partial Differential Equations

Next we abandon the search for explicit formulas and instead rely on functional analysis and relatively easy "energy" estimates to prove the existence of weak solutions to various linear PDE. We investigate also the uniqueness and regularity of such solutions, and deduce various other properties.

Chapter 5 is an introduction to Sobolev spaces, the proper setting for the study of many linear and nonlinear partial differential equations via energy methods. This is a hard chapter, the real worth of which is only later revealed, and requires some basic knowledge of Lebesgue measure theory. However, the requirements are not really so great, and the review in Appendix E should suffice. In my opinion there is no particular advantage in considering only the Sobolev spaces with exponent $p = 2$, and indeed insisting upon this obscures the two central inequalities, those of Gagliardo–Nirenberg–Sobolev (§5.6.1) and of Morrey (§5.6.2).

In Chapter 6 we vastly generalize our knowledge of Laplace's equation to other second-order elliptic equations. Here we work through a rather complete treatment of existence, uniqueness and regularity theory for solutions, including the maximum principle, and also a reasonable introduction to the study of eigenvalues, including a discussion of the principal eigenvalue for nonselfadjoint operators.

Chapter 7 expands the energy methods to a variety of linear partial differential equations characterizing evolutions in time. We broaden our earlier investigation of the heat equation to general second-order parabolic PDE, and of the wave equation to general second-order hyperbolic PDE. We study as well linear first-order hyperbolic systems, with the aim of motivating the developments concerning nonlinear systems of conservation laws in Chapter 11. The concluding section 7.4 presents the alternative functional analytic method of semigroups for building solutions.

(Missing from this long Part II on linear partial differential equations is any discussion of distribution theory or potential theory. These are important topics, but for our purposes seem dispensable, even in a book of such length. These omissions do not slow us up much, and make room for more nonlinear theory.)

PART III: Theory for Nonlinear Partial Differential Equations

This section parallels for nonlinear PDE the development in Part II, but is far less unified in its approach, as the various types of nonlinearity must be treated in quite different ways.

Chapter 8 commences the general study of nonlinear partial differential equations with an extensive discussion of the calculus of variations. Here we set forth a careful derivation of the direct method for deducing the existence of minimizers, and discuss also a variety of variational systems and constrained problems, as well as minimax methods. Variational theory is the most useful and accessible of the methods for nonlinear PDE, and so this chapter is fundamental.

Chapter 9 is, rather like Chapter 4 before, a gathering of assorted other techniques of use for nonlinear elliptic and parabolic partial differential equations. We encounter here monotonicity and fixed-point methods, and a variety of other devices, mostly involving the maximum principle. We study as well certain nice aspects of nonlinear semigroup theory, to complement the linear semigroup theory from Chapter 7.

Chapter 10 is an introduction to the modern theory of Hamilton–Jacobi PDE, and in particular to the notion of "viscosity solutions". We encounter also the connections with the optimal control of ODE, through dynamic programming.

Chapter 11 picks up from Chapter 3 the discussion of conservation laws, now systems of conservation laws. Unlike the general theoretical developments in Chapters 5–9, for which Sobolev spaces provide the proper abstract framework, we are forced to employ here direct linear algebra and calculus computations. We pay particular attention to the solution of Riemann's problem and to entropy criteria.

Appendices A–E provide for the reader's convenience some background material, with selected proofs, on inequalities, linear functional analysis, measure theory, etc.

The Bibliography primarily provides a listing of interesting PDE books to consult for further information. Since this is a textbook , and not a reference monograph, I have mostly not attempted to track down and document the original sources for the myriads of ideas and methods we will encounter. The mathematical literature for partial differential equations is truly vast, but the books cited in the Bibliography should at least provide a starting point for locating the primary sources.

1.5. PROBLEMS

1. Classify each of the partial differential equations in §1.2 as follows:
   (a) Is the PDE linear, semilinear, quasilinear or fully nonlinear?
   (b) What is the order of the PDE?

The next exercises provide some practice with the multiindex notation introduced in Appendix A.

2. Prove the *Multinomial Theorem*

$$(x_1 + \ldots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha,$$

where $\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha!}$, $\alpha! = \alpha_1!\alpha_2!\ldots\alpha_n!$, and $x^\alpha = x_1^{\alpha_1}\ldots x_n^{\alpha_n}$. The sum is taken over all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = k$.

3. Prove *Leibniz' formula*

$$D^\alpha(uv) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v,$$

where $u, v : \mathbb{R}^n \to \mathbb{R}$ are smooth, $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$, and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ $(i = 1, \ldots, n)$.

4. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is smooth. Prove

$$f(x) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}) \quad \text{as } x \to 0$$

for each $k = 1, 2, \ldots$. This is *Taylor's formula* in multiindex notation. (Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$.)