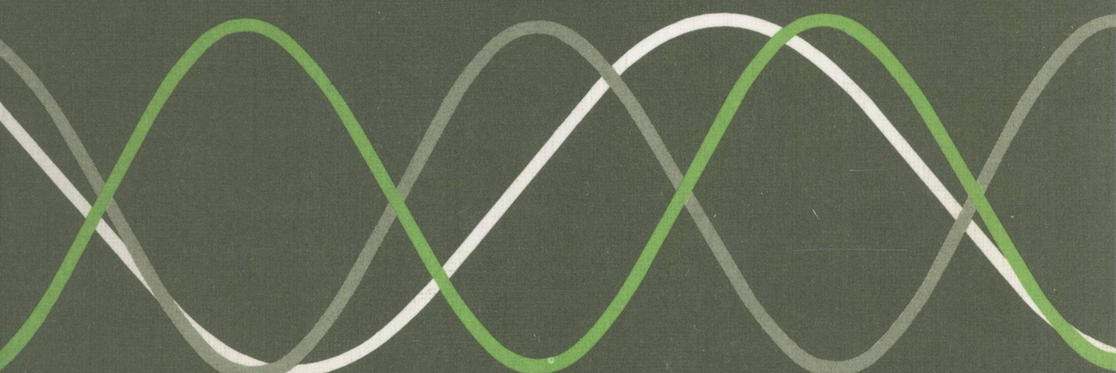
Fourier Series and Harmonic Analysis



K.A. Stroud

FOURIER SERIES and HARMONIC ANALYSIS

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Fourier Series and Harmonic Analysis

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PREFACE

A knowledge of Fourier series and of the techniques of Fourier analysis is now regarded as of utmost importance in the study of many branches of science and technology, for it provides a powerful tool in the solution of problems of a periodic nature occurring in situations involving electrical and mechanical vibrations, propagation of electromagnetic waves, acoustics, heat conduction, physical fields and the like.

The discovery by Jean Baptiste Joseph Fourier (1768–1830) that all periodic functions can be expressed as a sum of sinusoidal components permits two interpretations of the series representation of such a function:

(a) as an approximation to the function, where normally the first few terms of the infinite series suffice for practical purposes,
(b) as an analysis of the function into its harmonic components.

Both aspects are emphasised in the early chapters of the book.

An important advantage of the Fourier series representation of a function over the corresponding Taylor series representation is that the former can equally well represent a periodic function which contains a number of finite discontinuities, whereas Taylor series requires the use of successive differential coefficients.

Fourier series is a topic appearing in numerous course syllabuses. In some, the subject is pursued as far as the numerical harmonic analysis of a given waveform. This has wide applications and the subject is accordingly developed in detail in the first six chapters, which include also treatment of sine and cosine series representation of odd and even functions, and half-range series.

For those students wishing to proceed to rather wider aspects of the subject, Chapters 7 to 10 cover further applications and techniques, leading to the use of Fourier series in the solution of boundary value problems, Fourier integrals and an introduction to Fourier transforms.

Full mathematical rigour has not been attempted in a book of this size, but sufficient proofs have been included to establish the relevant results and to provide a foundation for the techniques used. Throughout the text, numerous worked examples are provided at each stage, together with sets of graded exercises by which the necessary practice can be undertaken and confidence with the methods assured. A complete set of answers is provided at the back of the book.

The author wishes to record his sincere thanks to all those who have shown an interest in the work and who have offered constructive comment; to acknowledge the many sources from which examples have been gleaned over the years; and to thank the publishers for their valuable advice in the preparation of the text for publication.

K.A.S.

CONTENTS

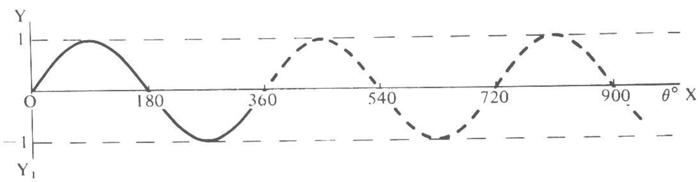
Chapter 1

PERIODIC FUNCTIONS

1.1 GRAPHS OF PERIODIC FUNCTIONS

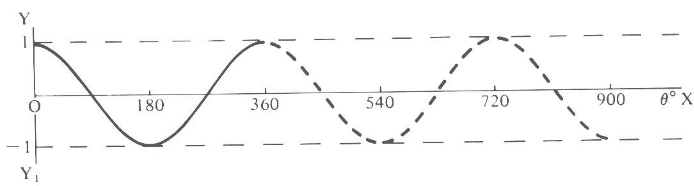
1.1.1 Characteristics

A *periodic function* is one in which the whole set of function values repeats at regular intervals of the independent variable. A common example of such a function is $\sin\theta$, which goes through its complete range of values as the angle θ increases from 0° to 360°. As θ continues past 360°, all the values of the function repeat, giving the characteristic waveform to the graph of $y = \sin\theta$.



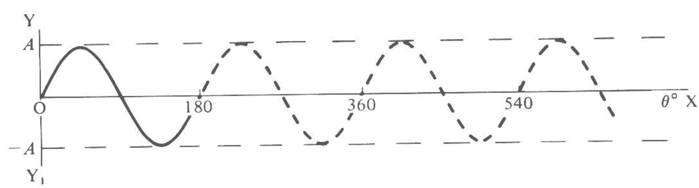
The constant interval of θ, after which repetition occurs, is called the *period* of the function and the part of the waveform extending over one period is referred to as *one cycle.*

The graph of $y = \cos\theta$ also illustrates a periodic function.

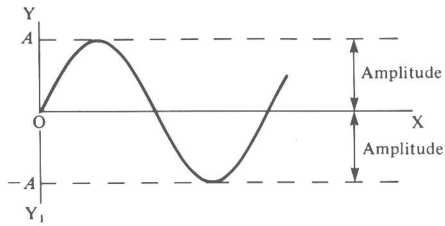


1.1.2 Graphs of $y = A\sin n\theta$

The function $y = A\sin 2\theta$ assumes all its values as the angle 2θ increases from 0° to 360°, i.e. as θ increases from 0° to 180°. Thus, one complete cycle occurs between $\theta = 0°$ and $\theta = 180°$.



The factor A denotes the *amplitude* of the function, i.e. the maximum displacement of the curve from its mean value.



Therefore, for $y = A\sin 2\theta$, the period is given by $2\theta = 360°$, i.e. $\theta = 180°$.
Similarly, for $y = A\sin 3\theta$, the period is given by $3\theta = 360°$, i.e. $\theta = 120°$.
and for $y = A\sin 4\theta$, the period is given by $4\theta = 360°$, i.e. $\theta = 90°$,
etc.
In general, for $y = A\sin n\theta$, the period is given by $n\theta = 360°$, i.e. $\theta = \dfrac{360°}{n}$.

Exercise 1

In each of the following cases, sketch the graph of the function over two cycles, indicating (i) the value of the amplitude and (ii) the values of θ at which the graph crosses the axis of θ.

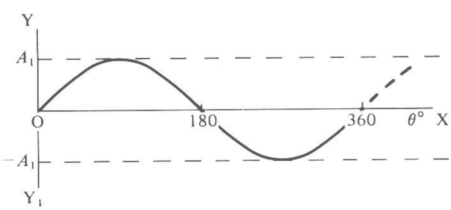
1. $y = 3\sin\theta$
2. $y = 2\sin 2\theta$
3. $y = \sin 4\theta$
4. $y = 5\cos 3\theta$
5. $y = 2\cos\theta$
6. $y = 4\sin 5\theta$
7. $y = 2\sin\dfrac{\theta}{2}$
8. $y = 3\sin 6\theta$
9. $y = \cos 2\theta$
10. $y = 5\cos 0.5\theta$

1.2 HARMONICS

1.2.1 Harmonics of $y = A\sin\theta$

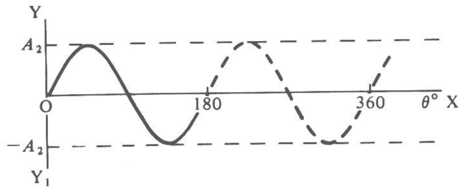
Consider again the graphs of $y = A_n \sin n\theta$, where n is a positive integer.

(a) $n = 1$, i.e. $y = A_1\sin\theta$



Period = 360°.
1 complete cycle in 360°.
This is the *fundamental* or *first harmonic* of $y = A\sin\theta$.

(b) $n = 2$, i.e. $y = A_2 \sin 2\theta$

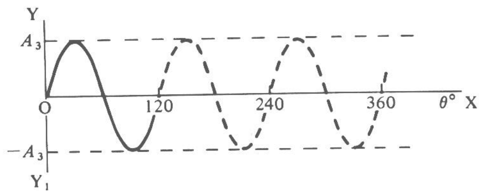


$$\text{Period} = \frac{360°}{2} = 180°.$$

2 complete cycles in 360°.

This is the *second harmonic* of $y = A \sin \theta$.

(c) $n = 3$, i.e. $y = A_3 \sin 3\theta$

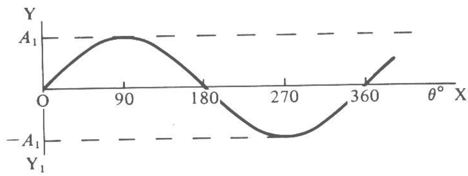


$$\text{Period} = \frac{360°}{3} = 120°.$$

3 complete cycles in 360°.

This is the *third harmonic* of $y = A \sin \theta$.

Therefore, we have

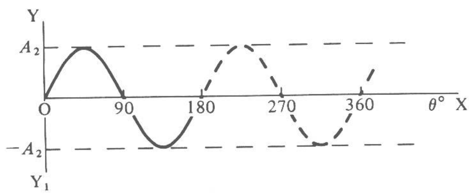


$y = A_1 \sin \theta$

Fundamental or first harmonic.

Period = 360°.

1 cycle in 360°.

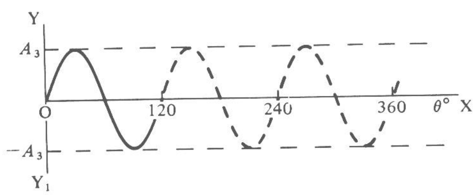


$y = A_2 \sin 2\theta$

Second harmonic.

$$\text{Period} = \frac{360°}{2} = 180°.$$

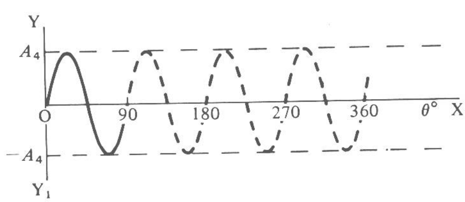
2 cycles in 360°.



$y = A_3 \sin 3\theta$

Third harmonic.

$$\text{Period} = \frac{360°}{3} = 120°.$$



$y = A_4 \sin 4\theta$

Fourth harmonic.

$$\text{Period} = \frac{360°}{4} = 90°.$$

4 cycles in 360°.

In general, the graph of $y = A_n \sin n\theta$ represents the nth *harmonic* of $y = A \sin\theta$. The period $= \dfrac{360°}{n}$ and there are thus n complete cycles in 360° or 2π radians.

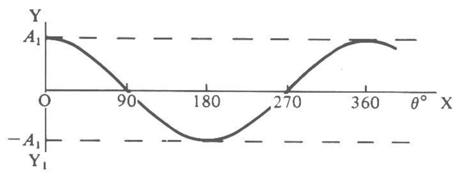
1.2.2 Odd and even harmonics

$\sin 2\theta, \sin 4\theta, \sin 6\theta, \ldots,$ are *even* harmonics of $\sin\theta$.

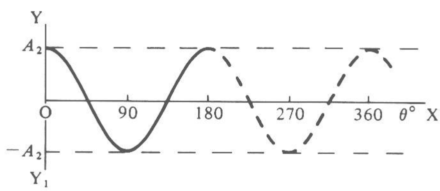
$\sin 3\theta, \sin 5\theta, \sin 7\theta, \ldots,$ are *odd* harmonics of $\sin\theta$.

So, with n an even integer, $y = A_n \sin n\theta$ indicates even harmonics and with n an odd integer, $y = A_n \sin n\theta$ indicates odd harmonics.

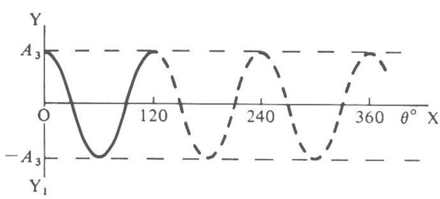
1.2.3 Graphs of $y = A_n \cos n\theta$



$y = A_1 \cos\theta$
Fundamental or first harmonic.
Period = 360°.
1 cycle in 360°.



$y = A_2 \cos 2\theta$
Second harmonic.
Period $= \dfrac{360°}{2} = 180°$.
2 cycles in 360°.



$y = A_3 \cos 3\theta$
Third harmonic.
Period $= \dfrac{360°}{3} = 120°$.
3 cycles in 360°.

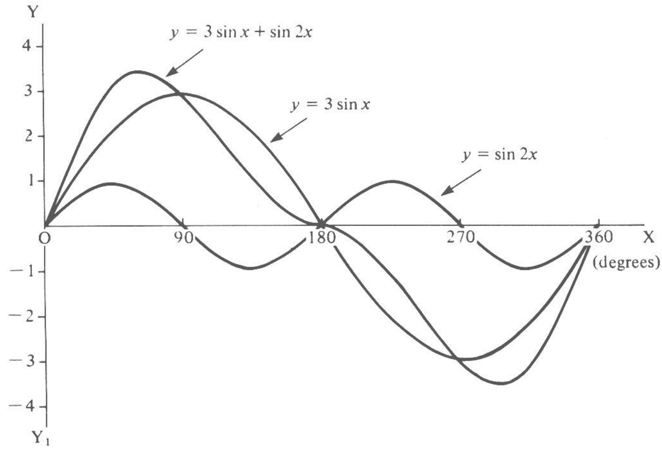
In general, the graph of $y = A_n \cos n\theta$ represents the nth harmonic of $y = A\cos\theta$. The period $= \dfrac{360°}{n}$ and there are thus n complete cycles in 360° or 2π radians.

1.3 COMPOUND WAVEFORMS

Compound waveforms can result from the addition of two or more sine or cosine curves, often a fundamental and one or more harmonics.

Example 1

To obtain the graph of $y = 3\sin x + \sin 2x$, we can plot the graphs of $y = 3\sin x$ and $y = \sin 2x$ separately and add the ordinates.



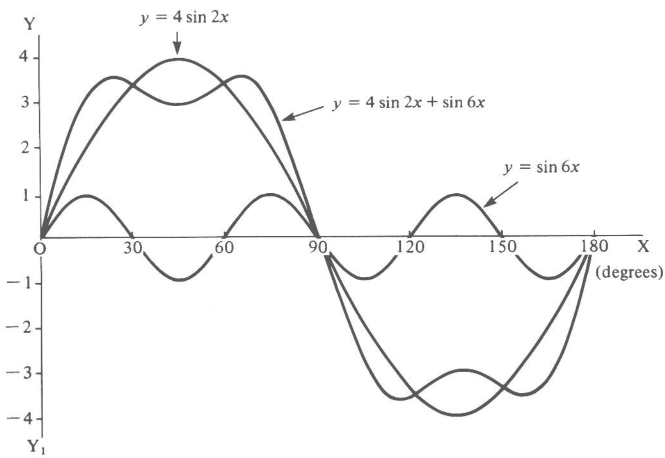
Note that the period of $y = 3\sin x$ is 360° and that of $y = \sin 2x$ is 180°.

Therefore, there are two cycles of $y = \sin 2x$ in the one cycle of $y = 3\sin x$.

The period of the combined function, $y = 3\sin x + \sin 2x$, is 360°, which is the period of its fundamental component.

Example 2

To obtain the graph of $y = 4\sin 2x + \sin 6x$, we proceed very much as before.



In this case, $y = \sin 6x$ is the third harmonic of $y = 4\sin 2x$.

The period of $y = 4\sin 2x$ is 180°; the period of $y = \sin 6x$ is 60°.

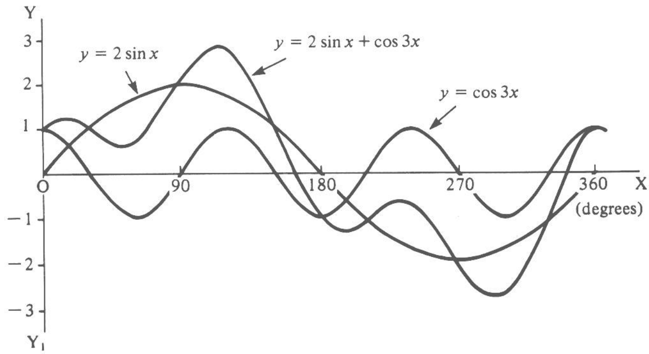
Therefore, there are three complete cycles of $y = \sin 6x$ in one complete cycle of $y = 4\sin 2x$.

The period of $y = 4\sin 2x + \sin 6x$ is therefore 180°.

Example 3

To obtain the graph of $y = 2\sin x + \cos 3x$.

Before we plot the curves, we know that the period of $y = 2\sin x$ is 360° and that the period of $y = \cos 3x$ is 120°. Therefore there are three complete cycles of $y = \cos 3x$ in one complete cycle of $y = 2\sin x$.



Many technological situations give rise to compound periodic waveforms and, in order to study one such output, it is convenient to analyse it into a number of component sine and cosine constituents. Such a process is therefore the reverse of that employed in the three examples worked above. Just how this analysis is achieved is explained in detail in a later chapter.

Exercise 2

Without drawing any graphs, state the period of each of the following functions.

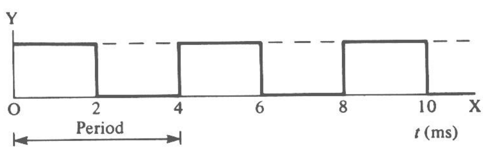
1. $y = 4\sin 2x$
2. $y = 2\cos 5x$
3. $y = 3\sin \dfrac{x}{2}$
4. $y = \sin 4x$
5. $y = 5\cos 3x$
6. $y = 2\sin \dfrac{3x}{5}$
7. $y = 5\sin x$
8. $y = 4\sin x + 3\sin 2x$
9. $y = 2\sin x + \cos 3x$
10. $y = 6\sin 2x + 2\sin 4x$

1.4 NON-SINUSOIDAL PERIODIC FUNCTIONS

Not all periodic functions are sinusoidal in appearance.

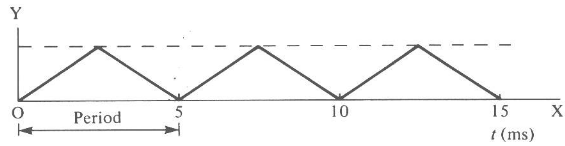
Examples

1.



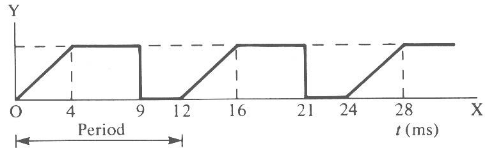
Period = 4 ms

2.



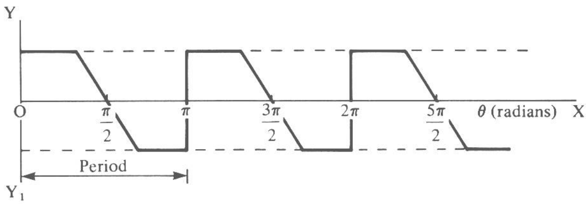
Period = 5 ms

3.



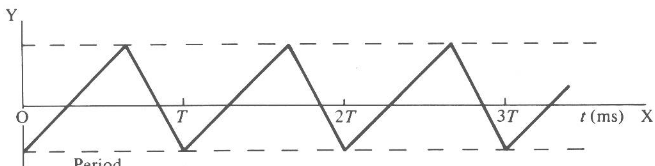
Period = 12 ms

4.



Period = π radians

In each case, the period is the shortest interval of the independent variable before repetition occurs.

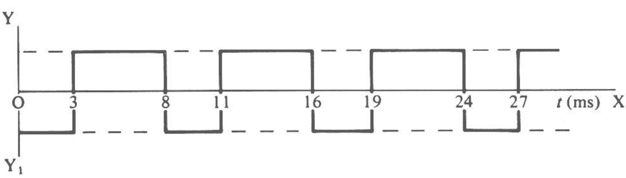


Period = T milliseconds

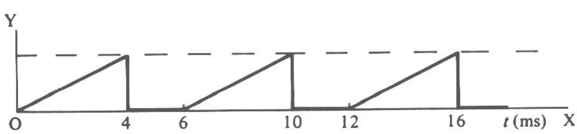
Exercise 3

In each of the following cases, the independent variable is time in milliseconds. Write down the period of each waveform shown.

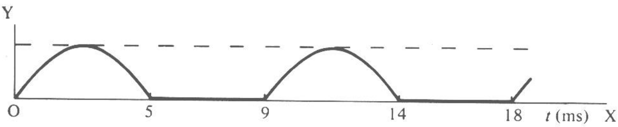
1.



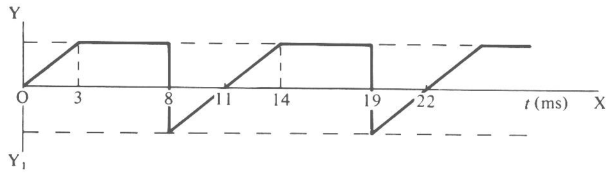
2.



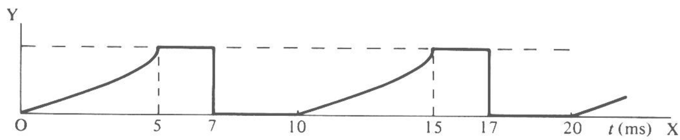
3.



4.



5.

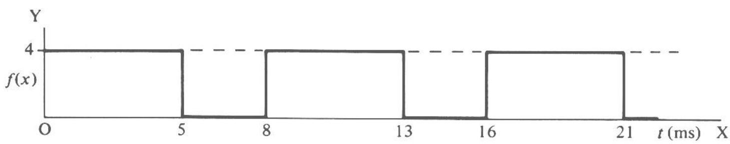


1.5 ANALYTICAL DESCRIPTION OF A PERIODIC FUNCTION

1.5.1 Given waveforms

It is often convenient to describe a periodic function algebraically.

Example 1



We see that

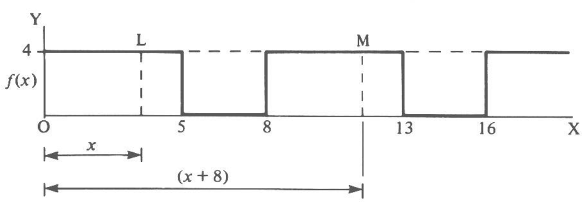
(a) between $x = 0$ and $x = 5$, the graph is the line $y = 4$. This can be written

$$f(x) = 4 \qquad 0 < x < 5$$

(b) between $x = 5$ and $x = 8$, the graph is the line $y = 0$,

i.e. $$f(x) = 0 \qquad 5 < x < 8$$

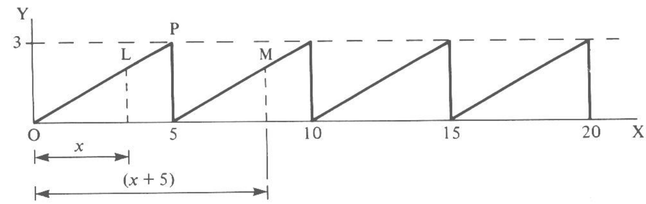
(c) the period of the function is 8 units.



Therefore, we can completely define the function in the form

$$f(x) = 4 \qquad 0 < x < 5$$
$$f(x) = 0 \qquad 5 < x < 8$$
$$f(x) = f(x+8)$$

Example 2



The equation of OP is $y = \frac{3}{5}x$.

(a) Between $x = 0$ and $x = 5$, the graph is the line $y = \frac{3}{5}x$,

i.e. $$f(x) = \tfrac{3}{5}x \qquad 0 < x < 5$$

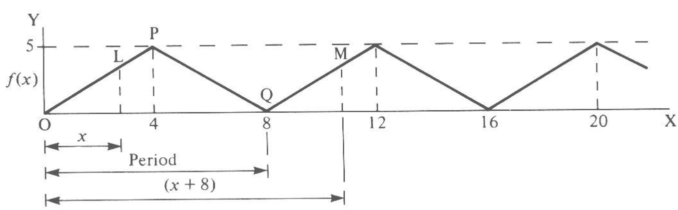
(b) The period of the function is 5,

i.e. $$f(x) = f(x+5)$$

Therefore, the periodic function can be defined as

$$f(x) = \tfrac{3}{5}x \qquad 0 < x < 5$$
$$f(x) = f(x+5)$$

Example 3



(a) The equation of OP is $y = \frac{5}{4}x$.

$$\therefore\ f(x) = \tfrac{5}{4}x, \qquad 0 < x < 4$$

(b) The equation of PQ is $y = -\frac{5}{4}x + 10$.

$$\therefore\ f(x) = -\tfrac{5}{4}x + 10, \qquad 4 < x < 8$$

(c) Period = 8

$$\therefore\ f(x) = f(x+8)$$