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B.A. Dubrovin
A.T. Fomenko
S.P. Novikov

Modern Geometry-Methods and Applications

Part III Introduction to Homology Theory

现代几何学方法和应用

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B. A. Dubrovin
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Part III. Introduction to Homology Theory

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B. A. Dubrovin
c/o VAAP—Copyright Agency of the U.S.S.R.
B. Bronnaya 6a
103104 Moscow
U.S.S.R.

S. P. Novikov
L. D. Landau Institute for Theoretical Physics
Academy of Sciences of the U.S.S.R.
Vorobevskoe Shosse, 2
117334 Moscow
U.S.S.R.

A. T. Fomenko
Department of Geometry and Topology
Faculty of Mathematics and Mechanics
Moscow State University
119899 Moscow
U.S.S.R.

R. G. Burns (*Translator*)
Department of Mathematics
York University Downsview
Ontario, M3J1P3
Canada

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Preface

In expositions of the elements of topology it is customary for homology to be given a fundamental role. Since Poincaré, who laid the foundations of topology, homology theory has been regarded as the appropriate primary basis for an introduction to the methods of algebraic topology. From homotopy theory, on the other hand, only the fundamental group and covering-space theory have traditionally been included among the basic initial concepts. Essentially all elementary classical textbooks of topology (the best of which is, in the opinion of the present authors, Seifert and Threlfall's *A Textbook of Topology*) begin with the homology theory of one or another class of complexes. Only at a later stage (and then still from a homological point of view) do fibre-space theory and the general problem of classifying homotopy classes of maps (homotopy theory) come in for consideration. However, methods developed in investigating the topology of differentiable manifolds, and intensively elaborated from the 1930s onwards (by Whitney and others), now permit a wholesale reorganization of the standard exposition of the fundamentals of modern topology. In this new approach, which resembles more that of classical analysis, these fundamentals turn out to consist primarily of the elementary theory of smooth manifolds,† homotopy theory based on these, and smooth fibre spaces. Furthermore, over the decade of the 1970s it became clear that exactly this complex of topological ideas and methods were proving to be fundamentally applicable in various areas of modern physics. It was for these reasons that the present authors regarded as absolutely

† Evidently the beginning ideas of topology, which can be traced back to Gauss, Riemann and Poincaré, actually arose, historically speaking, in this order. However, at the time of Gauss and Riemann, a correspondingly organized conceptual basis for a theory of topology was unrealizable. It was Poincaré who, in creating the homology theory of simplicial complexes, was able to provide a quite different, precise foundation for algebraic topology.

essential material for a training in topology, in the first place precisely the theory of smooth manifolds, homotopy theory, and fibre spaces, and incorporated this subject matter in Part II of their textbook *Modern Geometry*. It is assumed in the present text that the reader is acquainted with that material.

On the other hand, the solution of the more complex problems arising both within topology itself (the computation of homotopy groups, the classification of smooth manifolds, etc.) and in the numerous applications of the algebro-topological machinery to algebraic geometry and complex analysis, requires a very extensive elaboration of the methods of homology theory. There is in the contemporary topological literature a complete lack of books from which one might assimilate the complex of methods of homology theory useful in applications within topology. It is part of the aim of the present book to remedy this deficiency.

In expounding homology theory we have, wherever possible, striven to avoid using the abstract terminology of homological algebra, in order that the reader continually remain cognizant of the fact that cycles and boundaries, and homologies between them, are after all concrete geometrical objects. In a few places, for instance in the section devoted to spectral sequences, this self-imposed restriction has inevitably led to certain defects of exposition. However, it is our experience that the usual expositions of the machinery of modern homological algebra lead to worse defects in the reader's understanding, essentially because the geometric significance of the material is lost from view. Certain fundamental methods of modern algebraic topology (notably those associated with spectral sequences and cohomology operations) are described without full justification, since this would have required a substantial increase in the volume of material. It must be remembered that those methods are based exclusively on the formal algebraic properties of the algebraic entities with which they are concerned, and in no way involve their explicit geometric prototypes whence they derive their *raison d'être*. In the final chapter of the book the methods of algebraic topology are applied to the investigation of deep properties of characteristic classes and smooth structures on manifilds. It is the intention of the authors that the present monograph provide a path for the reader giving access to the contemporary topological literature.

A large contribution to the final version of this book was made by the editor, Victor Matveevich Bukhshtaber. Under his guidance several sections were rewritten, and many of the proofs improved upon. We thank him for carrying out this very considerable task.

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Contents

CHAPTER 1

Homology and Cohomology. Computational Recipes

§1. Cohomology Groups as Classes of Closed Differential Forms. Their Homotopy Invariance

Among the most important of the homotopy invariants of a manifold are its homology and cohomology groups, which we have already encountered (in §§19.3, 24.7, 25.5 of Part II), and which we shall now expound systematically.

There are several (equivalent) ways of defining the homology groups of a manifold; to begin with we give the definition (of the cohomology groups) in terms of differential forms on the manifold (as in §25.5 of Part II). Thus we shall initially be considering closed differential forms of rank k on our manifold M^n (where as usual the index n indicates the dimension of the manifold), given locally by

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad d\omega \equiv 0. \tag{1}$$

(Recall that a differential k-form is *closed* if $d\omega \equiv 0$, and is *exact* if $\omega = d\omega'$ for some form ω' of rank $k - 1$, and also that $d(d\omega') \equiv 0$, so that the exact forms figure among the closed ones (see §25.2 of Part I).)

1.1 Definition.† The *kth cohomology group* $H^k(M^n; \mathbb{R})$ (actually a real vector space) of a manifold M^n is the quotient group of the group (vector space) of all closed forms of rank k on M^n by its subgroup (linear subspace) of exact

† In the sequel we shall give several different definitions of the homology and cohomology groups with coefficients from various groups. In view of the fact that these definitions all yield essentially the same concept (see §§6, 14 below), we shall refrain from introducing indices to indicate any particular version of the concept as it arises in the various contexts.

forms. Thus the elements of $H^k(M^n; \mathbb{R})$ are the equivalence classes of closed k-forms where two forms are taken as equivalent if they differ by an exact form:

$$\omega_1 \sim \omega_2 \qquad \text{means} \qquad \omega_1 - \omega_2 = d\omega'. \tag{2}$$

The following result gives the simplest property of the (0th) cohomology groups.

1.2. Proposition. *For any manifold M^n the 0th cohomology group $H^0(M^n; \mathbb{R})$ is the vector space whose dimension q is equal to the number of connected components of the manifold.*

Proof. A form of rank zero is just an ordinary scalar function $f(x)$ on the manifold. If such a form is closed, then $df(x) \equiv 0$, so that $f(x)$ is locally constant, and therefore constant on each connected component of the manifold. Hence each closed 0-form on M^n can be identified with a sequence of q constants, one for each of the q components of the manifold. In view of the fact that there are no exact 0-forms, the proposition now follows. □

Any smooth map $f: M_1 \to M_2$ between manifolds determines a map $\omega \mapsto f^*(\omega)$, the "pullback", of forms ω on M_2 to forms $f^*(\omega)$ on M_1, satisfying $df^*(\omega) = f^*(d\omega)$ (see §§22.1, 25.2 of Part I). Hence each such map f determines a map (in fact a homomorphism, or better still a linear transformation)

$$f^*: H^k(M_2; \mathbb{R}) \to H^k(M_1; \mathbb{R}) \tag{3}$$

between the cohomology groups (since under f^* closed forms are sent to closed forms, and exact to exact).

1.3. Theorem. *Let $f_1: M_1 \to M_2$, $f_2: M_1 \to M_2$ be two smooth maps of manifolds. If f_1 is homotopic to f_2 then the corresponding homomorphisms f_1^* and f_2^* of the cohomology groups, coincide:*

$$f_1^* = f_2^*: H^k(M_2; \mathbb{R}) \to H^k(M_1; \mathbb{R}).$$

Proof. Let $F: M_1 \times I \to M_2$ be a smooth homotopy between f_1 and f_2, where I is the interval $1 \le t \le 2$, $F(x, 1) = f_1(x)$, and $F(x, 2) = f_2(x)$. In terms of local co-ordinates on $M_1 \times I$ of the form $(x^1, \ldots, x^n, t) = (x, t)$, where $x^1, \ldots, x^n$ are local co-ordinates on M_1, any differential form Ω of rank k on $M_1 \times I$ can be written as

$$\Omega = \omega_1 + \omega_2 \wedge dt, \qquad \Omega|_{t=t_0} = \omega_1(t_0), \tag{4}$$

where ω_1 is a form of rank k which does not involve the differential dt (in the sense that all of its components of the form

$$b_{i_1 \ldots i_{k-1}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \wedge dt, \qquad i_1 < \cdots < i_{k-1},$$

are identically zero), and ω_2 is a form of rank $(k-1)$ with the same property. Let ω be any form of rank k on the manifold M_2, and write $F^*(\omega) = \Omega =$

$\omega_1 + \omega_2 \wedge dt$, with ω_1 and ω_2 as just described, i.e. given locally by

$$\omega_2 = \sum_{i_1 < \cdots < i_{k-1}} a_{i_1 \ldots i_{k-1}}(x, t)\, dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}},$$

$$\omega_1 = \sum_{j_1 < \cdots < j_k} b_{j_1 \ldots j_k}(x, t)\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

We now define (locally) a form $D\Omega$ of rank $(k-1)$ on the manifold $M_1 \times I$, by means of the formula

$$D\Omega = \sum_{i_1 < \cdots < i_{k-1}} \left(\int_1^2 a_{i_1 \ldots i_{k-1}}(x, t)\, dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$$

$$= (-1)^{k-1} \int_1^2 \omega_2\, dt. \tag{5}$$

At this point we require a certain property of the form $D\Omega$, to establish which we now interrupt our proof.

1.4. Lemma. *The following formula holds* (*cf. the defining condition for an "algebraic homotopy" in* §2(5) *below*):

$$d(D(F^*(\omega))) \pm D(d(F^*(\omega))) = f_2^*(\omega) - f_1^*(\omega). \tag{6}$$

PROOF. We shall show that in fact for any form Ω on $M_1 \times I$, the following formula is valid:

$$dD(\Omega) \pm D(d\Omega) = \Omega|_{t=2} - \Omega|_{t=1}. \tag{7}$$

To this end we calculate $dD\Omega$ and $Dd\Omega$, with $\Omega = \omega_1 + \omega_2 \wedge dt$ as before. Locally we have (by definition of the operator d and its various properties—see §25.2 of Part I)

$$dD\Omega = \sum_{i_1 < \cdots < i_{k-1}} \sum_j \left(\int_1^2 \frac{\partial a_{i_1 \ldots i_{k-1}}}{\partial x^j}\, dt \right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}.$$

On the other hand

$$Dd\Omega = D(d\omega_1) + D(d\omega_2 \wedge dt)$$

$$= D\left(\sum_{j_1 < \cdots < j_k} \sum_q \frac{\partial b_{j_1 \ldots j_k}}{\partial x^q}\, dx^q \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right.$$

$$\left. + \sum_{j_1 < \cdots < j_k} \frac{\partial b_{j_1 \ldots j_k}}{\partial t}\, dt \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right)$$

$$+ D\left(\sum_{i_1 < \cdots < i_{k-1}} \sum_p \frac{\partial a_{i_1 \ldots i_{k-1}}}{\partial x^p}\, dx^p \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \wedge dt \right)$$

$$= \sum_{j_1 < \cdots < j_k} (b_{j_1 \ldots j_k}(x, 2) - b_{j_1 \ldots j_k}(x, 1))\, dx^{j_1} \wedge \cdots \wedge dx^{j_k} + (-1)^{k-1}\, dD\Omega$$

$$= \Omega|_{t=2} - \Omega|_{t=1} + (-1)^{k-1}\, dD\Omega,$$

whence the desired formula (7). Putting $\Omega = F^*(\omega)$, so that $\Omega|_{t=2} = f_2^*(\omega)$, $\Omega|_{t=1} = f_1^*(\omega)$, formula (7) then yields (6), completing the proof of the lemma. □

We now return to the proof of the theorem. Let ω be any closed form on M_2 (so that $d\omega \equiv 0$). Then, since $dF^*(\omega) = F^*(d\omega) \equiv 0$, formula (6) yields

$$dDF^*(\omega) = f_2^*(\omega) - f_1^*(\omega),$$

so that the difference of the forms $f_2^*(\omega)$ and $f_1^*(\omega)$ is exact. Since this is by definition equivalent to the statement that the homomorphisms f_1^*, $f_2^*: H^k(M_2; \mathbb{R}) \to H^k(M_1; \mathbb{R})$ coincide, the proof of the theorem is complete. □

Recall (from §17.4 of Part II) that two manifolds M_1, M_2 are said to be *homotopically equivalent* if there exist (smooth) maps $f: M_1 \to M_2$ and $g: M_2 \to M_1$, such that the composites $gf: M_1 \to M_1$ and $fg: M_2 \to M_2$ are homotopic to the respective identity maps

$$M_1 \to M_1 \quad (x \mapsto x), \qquad M_2 \to M_2 \quad (y \mapsto y).$$

(Thus, for example, Euclidean space $\mathbb{R}^n$, as also the disc

$$D^n = \left\{ \sum_{\alpha=1}^{n} (x^\alpha)^2 \leq R^2 \right\},$$

is homotopically equivalent to the one-point space, or what is equivalent, is *contractible* (over itself to a point), meaning that the identity map $\mathbb{R}^n \to \mathbb{R}^n$ $(x \mapsto x)$ is homotopic to a constant map $(\mathbb{R}^n \to \{0\})$.)

1.5. Theorem. *Homotopically equivalent manifolds have isomorphic cohomology groups.*

PROOF. Let M_1, M_2 be homotopically equivalent manifolds, and let $f: M_1 \to M_2$, $g: M_2 \to M_1$ be maps satisfying the defining conditions (see above) of homotopy equivalence. Consider the corresponding homomorphisms $f^*: H^k(M_2; \mathbb{R}) \to H^k(M_1; \mathbb{R})$ and $g^*: H^k(M_1; \mathbb{R}) \to H^k(M_2; \mathbb{R})$. Since the maps fg and gf are homotopic to the appropriate identity maps, it follows from Theorem 1.3 that the homomorphisms $(fg)^* = g^*f^*$ and $(gf)^* = f^*g^*$ are actually the corresponding identity homomorphisms:

$$1 = g^*f^*: H^k(M_2) \to H^k(M_2),$$

$$1 = f^*g^*: H^k(M_1) \to H^k(M_1).$$

Hence f^* and g^* are (mutually inverse) isomorphisms, and the theorem is proved. □

Remark. This theorem suggests a way of extending the definition of the cohomology groups to any topological space X with the property that there is a manifold M in which it can be embedded ($M \supset X$) which "contracts" to

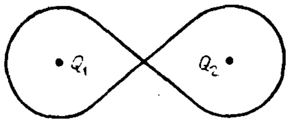

Figure 1

it, in the sense that the inclusion map $i\colon X \to M$ is a homotopy equivalence (so that there is a map $f\colon M \to X$ with the property that if and fi are homotopic to the appropriate identity maps). For such spaces X we simply define

$$H^k(X; \mathbb{R}) \equiv H^k(M; \mathbb{R}). \tag{8}$$

Thus, for instance, the "figure eight", while not a manifold, will now, according to this definition, have the same cohomology groups as $\mathbb{R}^2 \setminus \{Q_1, Q_2\}$, the plane with two points removed (see Figure 1).

1.6. Corollary. *The cohomology groups of Euclidean space $\mathbb{R}^n$ (and of the disc D^n) are isomorphic to those of a one-point space. Thus $H^k(\mathbb{R}^n)$ is trivial for $k > 0$, while $H^0(\mathbb{R}^n) \simeq \mathbb{R}$, the one-dimensional real vector space.*

This fact leads almost immediately to the so-called "Poincaré lemma": *Locally, i.e. in some neighbourhood of any point Q of a manifold M^n, every closed form ω ($d\omega \equiv 0$) of rank > 0 is exact: $\omega = d\omega'$.* To see this, we have merely to choose as the neighbourhood any disc $D^n = \{\sum_{\alpha=1}^{n} (x^\alpha - x_0^\alpha)^2 \leq \varepsilon\}$ with centre Q, wholly contained in some local co-ordinate neighbourhood (i.e. chart) of the manifold, and then apply the conclusion of Corollary 1.6, to the effect that $H^k(D^n) = 0$ for $k > 0$.)

The reader will no doubt recall the case $k = 1$ of the Poincaré lemma from courses in analysis: Given a 1-form $\omega = f_k\,dx^k$ with $d\omega \equiv 0$ (i.e. $\partial f_k/\partial x^i \equiv \partial f_i/\partial x^k$ in local notation), we have $\omega = dF$ where $F(P) = \int_Q^P f_k\,dx^k$, the (path-independent) line integral of the form along any smooth path in the disc from a fixed point Q to the variable point P.

What are the cohomology groups of the circle S^1?

1.7. Proposition. *The cohomology groups of the circle S^1 are as follows:*

$$\begin{gathered} H^k(S^1; \mathbb{R}) = 0 \quad \text{for} \quad k > 1; \\ H^1(S^1; \mathbb{R}) \simeq \mathbb{R}; \qquad H^0(S^1; \mathbb{R}) \simeq \mathbb{R}. \end{gathered} \tag{9}$$

PROOF. The triviality of the cohomology groups of S^1 for $k > 1$ is immediate from the fact that $\dim S^1 = 1$. That $H^0(S^1) \simeq \mathbb{R}$ follows from Proposition 1.2 and the connectedness of S^1. Thus we have only to show that $H^1(S^1) \simeq \mathbb{R}$.

To this end we introduce on S^1 the usual local co-ordinate φ, where for all integers n the numbers $\varphi + 2\pi n$ represent the same point of the circle as φ. A form of rank 1 is then given by $\omega = a(\varphi)\,d\varphi$, where $a(\varphi)$ is a periodic function on $\mathbb{R}$: $a(\varphi + 2\pi) = a(\varphi)$. We always have $d\omega = 0$, again since $\dim S^1 = 1$.

When will $\omega = a(\varphi)\,d\varphi$ be exact? Exactness in this context means precisely that $a(\varphi)\,d\varphi = dF$, where F is a periodic function, or equivalently that the function defined by

$$F(\varphi) = \int_0^{\varphi} a(\psi)\,d\psi + \text{const.}$$

is periodic of period 2π or, in yet other words, that $\int_{S^1} \omega = 0$.

We see therefore that a 1-form $\omega = a(\varphi)\,d\varphi$ on S^1 is exact precisely if $\int_{S^1} \omega = 0$, i.e. $\int_0^{2\pi} a(\varphi)\,d\varphi = 0$. Hence two 1-forms $\omega_1 = a(\varphi)\,d\varphi$ and $\omega_2 = b(\varphi)\,d\varphi$ determine the same cohomology class if and only if

$$\int_{S^1} \omega_1 = \int_{S^1} \omega_2, \qquad \text{i.e.} \qquad \int_0^{2\pi} a(\varphi)\,d\varphi = \int_0^{2\pi} b(\varphi)\,d\varphi,$$

so that the cohomology classes are in (appropriate) one-to-one correspondence with the possible values of such integrals, i.e. with $\mathbb{R}$. This completes the proof. □

1.8. Corollary. *The cohomology groups of the Euclidean plane with one point removed $\mathbb{R}^2 \backslash Q$ (or an annulus), being (by Theorem 1.5) isomorphic to those of a circle, are as follows:*

$$H^k(\mathbb{R}^2 \backslash Q) = 0, \quad k > 1; \qquad H^1(\mathbb{R}^2 \backslash Q) \simeq H^0(\mathbb{R}^2 \backslash Q) \simeq \mathbb{R}. \tag{10}$$

Remark. We indicate another method for calculating the first cohomology group $H^1(S^1)$ of the circle. With each 1-form $\omega(\varphi) = a(\varphi)\,d\varphi$ on the circle, we associate its *average* $\hat{\omega}$ (also a form) defined by

$$\hat{\omega} = \frac{1}{2\pi} \int_0^{2\pi} \omega(\varphi + \tau)\,d\tau = \frac{1}{2\pi} \left[\int_0^{2\pi} a(\varphi + \tau)\,d\tau \right] d\varphi.$$

1.9. Proposition. *The forms ω and $\hat{\omega}$ are cohomologous.*

PROOF. For each fixed τ the form $\omega(\varphi + \tau)$ is induced from ω via the map $\varphi + \tau \mapsto \varphi$ of the circle onto itself. Since such a map is homotopic to the identity, we have (by Theorem 1.3) that $\omega(\varphi) \sim \omega(\varphi + \tau)$. For an arbitrary Riemann sum for the form $\hat{\omega}$ (as an integral) we shall therefore have

$$\frac{1}{2\pi} \sum_i \omega(\varphi + \tau_i)\Delta\tau_i \sim \omega(\varphi) \cdot \frac{1}{2\pi} \sum_i \Delta\tau_i = \omega(\varphi). \tag{11}$$

Since any Riemann sum for $\hat{\omega}$ is thus cohomologous to ω, it follows that $\hat{\omega}$ will also be cohomologous to ω, as required. □

Continuing with our remark, we note next that $\hat{\omega}$ is given by

$$\hat{\omega}(\varphi) = \alpha\,d\varphi, \qquad \text{where} \quad \alpha = \text{const.} = \frac{1}{2\pi} \int_0^{2\pi} a(\psi)\,d\psi,$$

since

$$\hat{\omega}(\varphi) = \frac{1}{2\pi}\left[\int_0^{2\pi} a(\varphi + \tau)\,d\tau\right] d\varphi = \frac{1}{2\pi}\left[\int_\varphi^{2\pi+\varphi} a(\psi)\,d\psi\right] d\varphi$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} a(\psi)\,d\psi\right] d\varphi.$$

(Thus the form $\hat{\omega}(\varphi)$ is, as they say, "rotation-invariant": $\hat{\omega}(\varphi + \varphi_0) = \hat{\omega}(\varphi)$.) From this and the above proposition, we see that the correspondence $\omega \mapsto \hat{\omega}$ essentially associates (in what is clearly an appropriate one-to-one manner) a real number, namely α, with each 1-form ω on the circle, whence $H^1(S^1) \simeq \mathbb{R}$. In the sequel we shall use a generalization of this method to calculate the cohomology groups of compact homogeneous spaces.

1.10. Proposition. *An orientable, closed, Riemannian manifold M^n of dimension n has non-trivial nth cohomology group $H^n(M^n)$.*

PROOF. As usual we denote by Ω the volume element on M; thus locally

$$\Omega = \sqrt{|g|}\,dx^1 \wedge \cdots \wedge dx^n,$$

where $g = \det(g_{ij})$, (g_{ij}) being the Riemannian metric with which we are assuming our manifold endowed. If the local co-ordinates on the charts of M^n are all arranged to agree in orientation (i.e. so that the Jacobians of the transition functions on the regions of overlap are all positive), then (see Part I, §18.2) Ω can be regarded as a differential form of rank n on M^n, which can therefore be integrated over M, yielding its volume $\int_{M^n} \Omega > 0$.

Since M^n has dimension n and $d\Omega$ has rank $n + 1$, we must of course have $d\Omega = 0$, i.e. Ω is closed. If Ω were in fact exact, say $\Omega = d\omega$, then by the general Stokes formula (Part I, §26.3) we should have

$$\int_{M^n} \Omega = \int_{M^n} d\omega = \int_{\partial M^n} \omega = 0, \tag{12}$$

since by hypothesis M^n is without boundary. Hence we have found a closed n-form which is not exact (namely Ω), whence the proposition. □

Remark. It will be shown below (in §3) that on the other hand for every non-orientable closed manifold M^n (for example, $M^2 = \mathbb{R}P^2$, the projective plane) the group $H^n(M^n; \mathbb{R})$ is trivial. (Of course, the above proof fails for such manifolds since the volume element does not behave like a differential form under co-ordinate changes with negative Jacobian.)

For any manifold M^n we write

$$H^*(M) = \sum_{k=0}^{n} H^k(M^n), \tag{13}$$

the direct sum of (all) the cohomology groups of M. The following proposition shows that the wedge (or exterior) product of forms can be used to define a "multiplicative" operation on $H^*(M)$, thereby turning it into a ring.

1.11. Proposition. *For any closed forms ω_1, ω_2 on M^n, the forms $\omega_1 \wedge \omega_2$ and $(\omega_1 + d\omega') \wedge \omega_2$ are also closed, and moreover cohomologous.*

PROOF. By Leibniz' formula (see Part I, Theorem 25.2.4) we have

$$d(\omega' \wedge \omega_2) = d\omega' \wedge \omega_2 \pm \omega' \wedge d\omega_2 = d\omega' \wedge \omega_2. \tag{14}$$

Hence

$$(\omega_1 + d\omega') \wedge \omega_2 = \omega_1 \wedge \omega_2 + d(\omega' \wedge \omega_2),$$

so that $\omega_1 \wedge \omega_2$ and $(\omega_1 + d\omega') \wedge \omega_2$ are cohomologous, as required. (The closure of $\omega_1 \wedge \omega_2$ is immediate from Leibniz' formula.) □

In view of this proposition the exterior-product operation on $H^*(M)$ is well defined. It is easy to see that with this as its multiplicative operation $H^*(M)$ becomes a ring (in fact, an algebra), called the *cohomology ring* of the manifold M^n. Note that if $\omega_1 \in H^p(M^n)$, $\omega_2 \in H^q(M^n)$, then $\omega_1\omega_2 \in H^{p+q}(M^n)$, and that the multiplication in $H^*(M)$ is skew-commutative in the sense that (see Part I, Lemma 18.3.1)

$$\omega_2\omega_1 = (-1)^{pq}\omega_1\omega_2. \tag{15}$$

We shall now describe the geometric significance of the cohomology groups. (More precise considerations will be left to later sections.)

Given any manifold M^n we define "periods", or "integrals over cycles", of any closed form ω (of rank k) on M^n, as follows. As a preliminary, we define a *cycle* in M^n to be a pair (M^k, f), where M^k is any k-dimensional manifold (of dimension equal to the rank of ω) and $f\colon M^k \to M^n$ is any smooth map.

1.12. Definition. The *period* of a k-form ω on M^n with respect to a cycle (M^k, f) is the integral $\int_{M^k} f^*(\omega)$.

Let N^{k+1} be any oriented manifold-with-boundary. Its boundary $\partial N^{k+1} = M^k$ say, is then a closed, oriented manifold (which may have several connected components). We define a *film* (see Appendix 2 for an explanation of this name) to be a map $F\colon N^{k+1} \to M^n$ from the manifold-with-boundary N^{k+1} to the manifold M^n under consideration.

1.13. Theorem

(i) *The period of an exact form ω on M^n with respect to any cycle (M^k, f) is zero.*

(ii) *The period of a closed form ω on M^n is zero with respect to any cycle (M^k, f) in M^n which is the boundary of a film (N^{k+1}, F) (i.e. is such that $M^k = \partial N^{k+1}$ and $F|_{M^k} = f$).*

PROOF. (i) Writing $\omega = d\omega'$, we have by the general Stokes formula

$$\int_{M^k} f^*\omega = \int_{M^k} f^*(d\omega') = \int_{M^k} d(f^*\omega') = \int_{\partial M^k} f^*\omega' = 0, \tag{16}$$

where the last equality is a consequence of the fact that the manifold M^k is without boundary.

(ii) Since M^k is the boundary of N^{k+1} (with orientation induced from that of N^{k+1}), and $F|_{M^k} = f$, the general Stokes formula yields

$$\int_{M^k} f^*\omega = \int_{N^{k+1}} dF^*(\omega) = \int_{N^{k+1}} F^*(d\omega) = 0, \tag{17}$$

where in the last equality we have used the hypothesis $d\omega = 0$. □

We note without proof the following important fact (a partial converse to part (i) of the above theorem): *If the period of a closed form is zero with respect to every cycle, then the form is exact.* (See §14 below.)

Example. *For the n-dimensional sphere S^n we have $H^k(S^n) = 0$ for $k \neq 0, n$.*

PROOF. For $k > n$ it is trivial that $H^k(S^n) = 0$, so we may assume $0 < k < n$. If (M^k, f) is any cycle in S^n (where $0 < k < n$), then by Sard's theorem (Theorem 10.2.1 of Part II), there are certainly points of S^n outside $f(M^k)$. If $Q \in S^n$ is such a point, then the cycle (M^k, f) may be regarded as a cycle in $S^n \setminus Q \cong \mathbb{R}^n$. Now, essentially by Poincaré's lemma (see above), every closed form on $\mathbb{R}^n$ is exact, so that by Theorem 1.13(i) the period of every closed k-form with respect to the cycle (M^k, f) is zero. Since the cycle (M^k, f) was arbitrary, it follows from the above-mentioned partial converse of Theorem 1.13(i) that every closed k-form on S^n is exact, whence (for $0 < k < n$) $H^k(S^n) = 0$. □

This fact can also be established by means of an argument analogous to that used above for calculating $H^1(S^1)$ (in the remark following Corollary 1.8): one first shows that each cohomology class of closed k-forms on S^n contains a form ω invariant under the group $SO(n+1)$ of (proper) isometries of S^n. Such a form is of course determined by its components at a single point of the sphere, and these components will be invariant under the stationary group $SO(n) \subset SO(n+1)$ fixing that point (i.e. under the stabilizer of that point). We leave it to the reader to deduce that if $0 < k < n$ then these components must all be zero. (Consider to begin with the case of a 1-form on $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ whose components at the origin of $\mathbb{R}^2$ are rotation-invariant.)

We shall now show how an analogue of this method can be used for calculating the cohomology groups of Lie groups and symmetric spaces.

Recall (from §6 of Part II) that a homogeneous space (see §5.1 of Part II) M of a Lie group G, with isotropy group H, is said to be *symmetric* if there is an involutory Lie automorphism of G, i.e. a Lie automorphism $I: G \to G$ such that $I^2 = 1$ and $I|_H = 1$ (so that the automorphism I fixes H pointwise); it is also required that all points fixed by I that are sufficiently close to the identity element of G, should lie in the subgroup H. Corresponding to each point x of such a manifold M there is then a naturally determined "symmetry" s_x of M, whose effect on an arbitrary point y of M is defined as follows: since