Graduate Texts in Mathematics

Saunders Mac Lane

Categories for the Working Mathematician

Second Edition

数学工作者必知的范畴学

第2版

Springer-Verlag

世界图书出版公司

Saunders Mac Lane

Categories for the Working Mathematician

Second Edition

Springer

书　　名：Categories for the Working Mathematician 2nd ed

作　　者：M. Lang

中 译 名：数学工作者必知的范畴学 第 2 版

出 版 者：世界图书出版公司北京公司

印 刷 者：北京世图印刷厂

发　　行：世界图书出版公司北京公司 (北京朝内大街 137 号　100010)

联系电话：010-64015659, 64038347

电子信箱：kjsk@vip.sina.com

开　　本：24　　印　张：14

出版年代：2003 年 6 月

书　　号：7-5062-6008-5 / O・397

版权登记：图字:01-2003-3999

定　　价：39.00 元

Graduate Texts in Mathematics 5

Springer
New York
Berlin
Heidelberg
Barcelona
Budapest
Hong Kong
London
Milan
Paris
Singapore
Tokyo

Saunders Mac Lane
Professor Emeritus
Department of Mathematics
University of Chicago
Chicago, IL 60637-1514
USA

Mathematics Subject Classification (1991): 18-01

Library of Congress Cataloging-in-Publication Data
Mac Lane, Saunders, 1909–
Categories for the working mathematician/Saunders Mac Lane. —
2nd ed.
p. cm. — (Graduate texts in mathematics; 5)
Includes bibliographical references and index.
ISBN 0-387-98403-8 (hardcover: alk. paper)
1. Categories (Mathematics) I. Title. II. Series.
QA169.M33 1998
512'.55—dc21 97-45229

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

ISBN 0-387-98403-8 Springer-Verlag New York Berlin Heidelberg SPIN 10661688

Preface to the Second Edition

This second edition of "Categories Work" adds two new chapters on topics of active interest. One is on symmetric monoidal categories and braided monoidal categories and the coherence theorems for them—items of interest in their own right and also in view of their use in string theory in quantum field theory. The second new chapter describes 2-categories and the higher-dimensional categories that have recently come into prominence. In addition, the bibliography has been expanded to cover some of the many other recent advances concerning categories.

The earlier 10 chapters have been lightly revised, clarifying a number of points, in many cases due to helpful suggestions from George Janelidze. In Chapter III, I have added a description of the colimits of representable functors, while Chapter IV now includes a brief description of characteristic functions of subsets and of the elementary topoi.

Dune Acres, March 27, 1997 Saunders Mac Lane

Preface to the First Edition

Category theory has developed rapidly. This book aims to present those ideas and methods that can now be effectively used by mathematicians working in a variety of other fields of mathematical research. This occurs at several levels. On the first level, categories provide a convenient conceptual language, based on the notions of category, functor, natural transformation, contravariance, and functor category. These notions are presented, with appropriate examples, in Chapters I and II. Next comes the fundamental idea of an adjoint pair of functors. This appears in many substantially equivalent forms: that of universal construction, that of direct and inverse limit, and that of pairs of functors with a natural isomorphism between corresponding sets of arrows. All of these forms, with their interrelations, are examined in Chapters III to V. The slogan is "Adjoint functors arise everywhere."

Alternatively, the fundamental notion of category theory is that of a monoid—a set with a binary operation of multiplication that is associative and that has a unit; a category itself can be regarded as a sort of generalized monoid. Chapters VI and VII explore this notion and its generalizations. Its close connection to pairs of adjoint functors illuminates the ideas of universal algebra and culminates in Beck's theorem characterizing categories of algebras; on the other hand, categories with a monoidal structure (given by a tensor product) lead inter alia to the study of more convenient categories of topological spaces.

Since a category consists of arrows, our subject could also be described as learning how to live without elements, using arrows instead. This line of thought, present from the start, comes to a focus in Chapter VIII, which covers the elementary theory of abelian categories and the means to prove all of the diagram lemmas without ever chasing an element around a diagram.

Finally, the basic notions of category theory are assembled in the last two chapters: more exigent properties of limits, especially of filtered limits; a calculus of "ends"; and the notion of Kan extensions. This is the deeper form of the basic constructions of adjoints. We end with the observations that all concepts of category theory are Kan extensions (§7 of Chapter X).

I have had many opportunities to lecture on the materials of these chapters: at Chicago; at Boulder, in a series of colloquium lectures to the American Mathematical Society; at St. Andrews, thanks to the Edinburgh Mathematical Society; at Zurich, thanks to Beno Eckmann and the Forschungsinstitut für Mathematik; at London, thanks to A. Fröhlich and Kings and Queens Colleges; at Heidelberg, thanks to H. Seifert and Albrecht Dold; at Canberra, thanks to Neumann, Neumann, and a Fulbright grant; at Bowdoin, thanks to Dan Christie and the National Science Foundation; at Tulane, thanks to Paul Mostert and the Ford Foundation; and again at Chicago, thanks ultimately to Robert Maynard Hutchins and Marshall Harvey Stone.

Many colleagues have helped my studies. I have profited much from a succession of visitors to Chicago (made possible by effective support from the Air Force Office of Scientific Research, the Office of Naval Research, and the National Science Foundation): M. André, J. Bénabou, E. Dubuc, F.W. Lawvere, and F.E.J. Linton. I have had good counsel from Michael Barr, John Gray, Myles Tierney, and Fritz Ulmer, and sage advice from Brian Abrahamson, Ronald Brown, W.H. Cockcroft, and Paul Halmos. Daniel Feigin and Geoffrey Phillips both managed to bring some of my lectures into effective written form. My old friend, A.H. Clifford, and others at Tulane were of great assistance. John MacDonald and Ross Street gave pertinent advice on several chapters; Spencer Dickson, S.A. Huq, and Miguel La Plaza gave a critical reading of other material. Peter May's trenchant advice vitally improved the emphasis and arrangement, and Max Kelly's eagle eye caught many soft spots in the final manuscript. I am grateful to Dorothy Mac Lane and Tere Shuman for typing, to Dorothy Mac Lane for preparing the index, and to M.K. Kwong for careful proofreading—but the errors that remain, and the choice of emphasis and arrangement, are mine.

Dune Acres, March 27, 1971 Saunders Mac Lane

Contents

Preface to the Second Edition v
Preface to the First Edition vii

Introduction 1

I. Categories, Functors, and Natural Transformations 7

1. Axioms for Categories 7
2. Categories 10
3. Functors 13
4. Natural Transformations 16
5. Monics, Epis, and Zeros 19
6. Foundations 21
7. Large Categories 24
8. Hom-Sets 27

II. Constructions on Categories 31

1. Duality 31
2. Contravariance and Opposites 33
3. Products of Categories 36
4. Functor Categories 40
5. The Category of All Categories 42
6. Comma Categories 45
7. Graphs and Free Categories 48
8. Quotient Categories 51

III. Universals and Limits 55

1. Universal Arrows 55
2. The Yoneda Lemma 59
3. Coproducts and Colimits 62
4. Products and Limits 68

5. Categories with Finite Products 72
6. Groups in Categories 75
7. Colimits of Representable Functors 76

IV. Adjoints 79

1. Adjunctions 79
2. Examples of Adjoints 86
3. Reflective Subcategories 90
4. Equivalence of Categories 92
5. Adjoints for Preorders 95
6. Cartesian Closed Categories 97
7. Transformations of Adjoints 99
8. Composition of Adjoints 103
9. Subsets and Characteristic Functions 105
10. Categories Like **Sets** 106

V. Limits 109

1. Creation of Limits 109
2. Limits by Products and Equalizers 112
3. Limits with Parameters 115
4. Preservation of Limits 116
5. Adjoints on Limits 118
6. Freyd's Adjoint Functor Theorem 120
7. Subobjects and Generators 126
8. The Special Adjoint Functor Theorem 128
9. Adjoints in Topology 132

VI. Monads and Algebras 137

1. Monads in a Category 137
2. Algebras for a Monad 139
3. The Comparison with Algebras 142
4. Words and Free Semigroups 144
5. Free Algebras for a Monad 147
6. Split Coequalizers 149
7. Beck's Theorem 151
8. Algebras Are T-Algebras 156
9. Compact Hausdorff Spaces 157

VII. Monoids 161

1. Monoidal Categories 161
2. Coherence 165

3. Monoids 170
4. Actions 174
5. The Simplicial Category 175
6. Monads and Homology 180
7. Closed Categories 184
8. Compactly Generated Spaces 185
9. Loops and Suspensions 188

VIII. Abelian Categories 191

1. Kernels and Cokernels 191
2. Additive Categories 194
3. Abelian Categories 198
4. Diagram Lemmas 202

IX. Special Limits 211

1. Filtered Limits 211
2. Interchange of Limits 214
3. Final Functors 217
4. Diagonal Naturality 218
5. Ends 222
6. Coends 226
7. Ends with Parameters 228
8. Iterated Ends and Limits 230

X. Kan Extensions 233

1. Adjoints and Limits 233
2. Weak Universality 235
3. The Kan Extension 236
4. Kan Extensions as Coends 240
5. Pointwise Kan Extensions 243
6. Density 245
7. All Concepts Are Kan Extensions 248

XI. Symmetry and Braiding in Monoidal Categories 251

1. Symmetric Monoidal Categories 251
2. Monoidal Functors 255
3. Strict Monoidal Categories 257
4. The Braid Groups B_n and the Braid Category 260
5. Braided Coherence 263
6. Perspectives 266

XII. Structures in Categories 267

1. Internal Categories 267
2. The Nerve of a Category 270
3. 2-Categories 272
4. Operations in 2-Categories 276
5. Single-Set Categories 279
6. Bicategories 281
7. Examples of Bicategories 283
8. Crossed Modules and Categories in **Grp** 285

Appendix. Foundations 289

Table of Standard Categories: Objects and Arrows 293

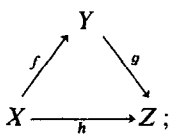
Table of Terminology 295

Bibliography 297

Index 303

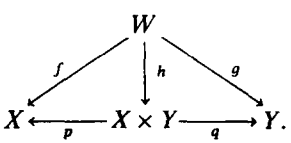
Introduction

Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams of arrows. Each arrow $f : X \to Y$ represents a function; that is, a set X, a set Y, and a rule $x \mapsto fx$ which assigns to each element $x \in X$ an element $fx \in Y$; whenever possible we write fx and not $f(x)$, omitting unnecessary parentheses. A typical diagram of sets and functions is



it is commutative when h is $h = g \circ f$, where $g \circ f$ is the usual composite function $g \circ f : X \to Z$, defined by $x \mapsto g(fx)$. The same diagrams apply in other mathematical contexts; thus in the "category" of all topological spaces, the letters X, Y, and Z represent topological spaces while f, g, and h stand for continuous maps. Again, in the "category" of all groups, X, Y, and Z stand for groups, f, g, and h for homomorphisms.

Many properties of mathematical constructions may be represented by universal properties of diagrams. Consider the cartesian product $X \times Y$ of two sets, consisting as usual of all ordered pairs $\langle x, y \rangle$ of elements $x \in X$ and $y \in Y$. The projections $\langle x, y \rangle \mapsto x$, $\langle x, y \rangle \mapsto y$ of the product on its "axes" X and Y are functions $p : X \times Y \to X$, $q : X \times Y \to Y$. Any function $h : W \to X \times Y$ from a third set W is uniquely determined by its composites $p \circ h$ and $q \circ h$. Conversely, given W and two functions f and g as in the diagram below, there is a unique function h which makes the diagram commute; namely, $hw = \langle fw, gw \rangle$ for each w in W:



Thus, given X and Y, $\langle p, q\rangle$ is "universal" among pairs of functions from some set to X and Y, because any other such pair $\langle f, g\rangle$ factors uniquely (via h) through the pair $\langle p, q\rangle$. This property describes the cartesian product $X \times Y$ uniquely (up to a bijection); the same diagram, read in the category of topological spaces or of groups, describes uniquely the cartesian product of spaces or the direct product of groups.

Adjointness is another expression for these universal properties. If we write $\hom(W, X)$ for the set of all functions $f: W \to X$ and $\hom(\langle U, V\rangle, \langle X, Y\rangle)$ for the set of all pairs of functions $f: U \to X$, $g: V \to Y$, the correspondence $h \mapsto \langle ph, qh\rangle = \langle f, g\rangle$ indicated in the diagram above is a bijection

$$\hom(W, X \times Y) \cong \hom(\langle W, W\rangle, \langle X, Y\rangle).$$

This bijection is "natural" in the sense (to be made more precise later) that it is defined in "the same way" for all sets W and for all pairs of sets $\langle X, Y\rangle$ (and it is likewise "natural" when interpreted for topological spaces or for groups). This natural bijection involves two constructions on sets: The construction $W \mapsto W, W$ which sends each set to the diagonal pair $\Delta W = \langle W, W\rangle$, and the construction $\langle X, Y\rangle \mapsto X \times Y$ which sends each pair of sets to its cartesian product. Given the bijection above, we say that the construction $X \times Y$ is a *right adjoint* to the construction Δ, and that Δ is left adjoint to the product. Adjoints, as we shall see, occur throughout mathematics.

The construction "cartesian product" is called a "functor" because it applies suitably to sets *and* to the functions between them; two functions $k: X \to X'$ and $l: Y \to Y'$ have a function $k \times l$ as their cartesian product:

$$k \times l: X \times Y \to X' \times Y', \quad \langle x, y\rangle \mapsto \langle kx, ly\rangle.$$

Observe also that the one-point set $1 = \{0\}$ serves as an identity under the operation "cartesian product", in view of the bijections

$$1 \times X \xrightarrow{\lambda} X \xleftarrow{\varrho} X \times 1 \tag{1}$$

given by $\lambda\langle 0, x\rangle = x$, $\varrho\langle x, 0\rangle = x$.

The notion of a monoid (a semigroup with identity) plays a central role in category theory. A monoid M may be described as a set M together with two functions

$$\mu: M \times M \to M, \quad \eta: 1 \to M \tag{2}$$

such that the following two diagrams in μ and η commute:

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{1 \times \mu} & M \times M \\ \downarrow{\scriptstyle \mu \times 1} & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow{\mu} & M, \end{array} \qquad \begin{array}{ccccc} 1 \times M & \xrightarrow{\eta \times 1} & M \times M & \xleftarrow{1 \times \eta} & M \times 1 \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \varrho} \\ M & = & M & = & M; \end{array} \tag{3}$$

here 1 in $1 \times \mu$ is the identity function $M \to M$, and 1 in $1 \times M$ is the one-point set $1 = \{0\}$, while λ and ϱ are the bijections of (1) above. To say that these diagrams commute means that the following composites are equal:

$$\mu \circ (1 \times \mu) = \mu \circ (\mu \times 1), \quad \mu \circ (\eta \times 1) = \lambda, \quad \mu \circ (1 \times \eta) = \varrho.$$

These diagrams may be rewritten with elements, writing the function μ (say) as a product $\mu(x, y) = xy$ for $x, y \in M$ and replacing the function η on the one-point set $1 = \{0\}$ by its (only) value, an element $\eta(0) = u \in M$. The diagrams above then become

$$\begin{array}{ccc} \langle x, y, z\rangle & \longmapsto & \langle x, yz\rangle \\ \downarrow & & \downarrow \\ \langle xy, z\rangle & \longmapsto & (xy)z = x(yz), \end{array} \qquad \begin{array}{ccc} \langle 0, x\rangle & \longmapsto & \langle u, x\rangle \\ \downarrow & & \downarrow \\ x & = & ux, \end{array} \qquad \begin{array}{ccc} \langle x, u\rangle & \longmapsto & \langle x, 0\rangle \\ \downarrow & & \downarrow \\ xu & = & x. \end{array}$$

They are exactly the familiar axioms on a monoid, that the multiplication be associative and have an element u as left and right identity. This indicates, conversely, how algebraic identities may be expressed by commutative diagrams. The same process applies to other identities; for example, one may describe a group as a monoid M equipped with a function $\zeta : M \to M$ (of course, the function $x \mapsto x^{-1}$) such that the following diagram commutes:

$$\begin{array}{ccccc} M & \xrightarrow{\delta} & M \times M & \xrightarrow{1 \times \zeta} & M \times M \\ \downarrow & & & & \downarrow{\scriptstyle \mu} \\ 1 & & \longrightarrow & & M \end{array} \qquad \begin{array}{ccccc} x & \longmapsto & \langle x, x\rangle & \longmapsto & \langle x, x^{-1}\rangle \\ \downarrow & & & & \downarrow \\ 0 & \longmapsto & u & = & xx^{-1}, \end{array} \tag{4}$$

here $\delta : M \to M \times M$ is the diagonal function $x \mapsto \langle x, x\rangle$ for $x \in M$, while the unnamed vertical arrow $M \to 1 = \{0\}$ is the evident (and unique) function from M to the one-point set. As indicated just to the right, this diagram does state that ζ assigns to each element $x \in M$ an element x^{-1} which is a right inverse to x.

This definition of a group by arrows μ, η, and ζ in such commutative diagrams makes no explicit mention of group elements, so applies to other circumstances. If the letter M stands for a topological space (not just a set) and the arrows are continuous maps (not just functions), then the conditions (3) and (4) define a topological group – for they specify that M is a topological space with a binary operation μ of multiplication which is continuous (simultaneously in its arguments) and which has a continuous right inverse, all satisfying the usual group axioms. Again, if the letter M stands for a differentiable manifold (of

class C^∞) while 1 is the one-point manifold and the arrows μ, η, and ζ are smooth mappings of manifolds, then the diagrams (3) and (4) become the definition of a Lie group. Thus groups, topological groups, and Lie groups can all be described as "diagrammatic" groups in the respective categories of sets, of topological spaces, and of differentiable manifolds.

This definition of a group in a category depended (for the inverse in (4)) on the diagonal map $\delta: M \to M \times M$ to the cartesian square $M \times M$. The definition of a monoid is more general, because the cartesian product $\times$ in $M \times M$ may be replaced by any other operation $\square$ on two objects which is associative and which has a unit 1 in the sense prescribed by the isomorphisms (1). We can then speak of a monoid in the system $(C, \square, 1)$, where C is the category, $\square$ is such an operation, and 1 is its unit. Consider, for example, a monoid M in $(\mathbf{Ab}, \otimes, \mathbf{Z})$, where **Ab** is the category of abelian groups, $\times$ is replaced by the usual tensor product of abelian groups, and 1 is replaced by $\mathbf{Z}$, the usual additive group of integers; then (1) is replaced by the familiar isomorphism

$$\mathbf{Z} \otimes X \cong X \cong X \otimes \mathbf{Z}, \qquad X \text{ an abelian group}.$$

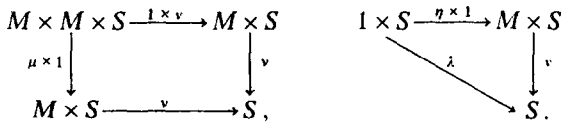
Then a monoid M in $(\mathbf{Ab}, \otimes, \mathbf{Z})$ is, we claim, simply a ring. For the given morphism $\mu: M \otimes M \to M$ is, by the definition of $\otimes$, just a function $M \times M \to M$, call it multiplication, which is bilinear; i.e., distributive over addition on the left and on the right, while the morphism $\eta: \mathbf{Z} \to M$ of abelian groups is completely determined by picking out one element of M; namely, the image u of the generator 1 of $\mathbf{Z}$. The commutative diagrams (3) now assert that the multiplication μ in the abelian group M is associative and has u as left and right unit – in other words, that M is indeed a ring (with identity = unit).

The (homo)-morphisms of an algebraic system can also be described by diagrams. If $\langle M, \mu, \eta \rangle$ and $\langle M', \mu', \eta' \rangle$ are two monoids, each described by diagrams as above, then a morphism of the first to the second may be defined as a function $f: M \to M'$ such that the following diagrams commute:

$$\begin{array}{ccc} M & M \times M \xrightarrow{\mu} M & 1 \xrightarrow{\eta} M \\ \downarrow f & \downarrow f \times f \quad \downarrow f & \| \quad \downarrow f \\ M', & M' \times M' \xrightarrow{\mu'} M', & 1 \xrightarrow{\eta'} M'. \end{array} \tag{5}$$

In terms of elements, this asserts that $f(xy) = (fx)(fy)$ and $fu = u'$, with u and u' the unit elements; thus a homomorphism is, as usual, just a function preserving composite and units. If M and M' are monoids in $(\mathbf{Ab}, \otimes, \mathbf{Z})$, that is, rings, then a homomorphism f as here defined is just a morphism of rings (preserving the units).

Finally, an *action* of a monoid $\langle M, \mu, \eta \rangle$ on a set S is defined to be a function $\nu : M \times S \to S$ such that the following two diagrams commute:



If we write $\nu(x, s) = x \cdot s$ to denote the result of the action of the monoid element x on the element $s \in S$, these diagrams state just that

$$x \cdot (y \cdot s) = (xy) \cdot s\,, \quad u \cdot s = s$$

for all $x, y \in M$ and all $s \in S$. These are the usual conditions for the action of a monoid on a set, familiar especially in the case of a group acting on a set as a group of transformations. If we shift from the category of sets to the category of topological spaces, we get the usual continuous action of a topological monoid M on a topological space S. If we take $\langle M, \mu, \eta \rangle$ to be a monoid in $(\mathbf{Ab}, \otimes, \mathbf{Z})$, then an action of M on an object S of $\mathbf{Ab}$ is just a left module S over the ring M.