LECTURE NOTES IN PHYSICS

S. S. Abdullaev

Construction of Mappings for Hamiltonian Systems and Their Applications

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Preface

Mappings constitute a powerful method for studying dynamical systems. They are fundamentally based on a formulation of dynamical equations governing them as a system of first-order ordinary differential equations. According to the theorem of Cauchy, the solutions of these dynamical equations are unique and are completely determined by the initial conditions, i.e., there exists the unique transformation or *mapping* of the initial conditions into the final conditions. The surface-to-surface maps (*Poincare return map*) and stroboscopic maps introduced by Poincaré (1892–99) replace the dynamics of a continuous system by a discrete one. These maps have important advantages in a study of dynamical systems. First, they reduce dimensions of the system at least by one. They allow one to visualize the dynamics of the system at certain sections (Poincaré sections) of phase space and thereby display the global behavior of the system. Many concepts of continuous systems become more clear when they are formulated using Poincaré maps. For instance, the study of stability of periodic orbits can be simply reduced to a study of stability of fixed points of the mappings.

The Hamiltonian formulation of dynamical equations of physical systems of different nature had a deep impact on the study of dynamical systems (Hamilton, 1834; Goldstein, 1980; Arnold, 1989). A system with N degrees of freedom can be described by $2N$ ordinary differential equations of first order in the phase space of the canonical coordinates $q = (q_1, \dots, q_N)$ and momenta $p = (p_1, \dots, p_N)$, and are determined by a single scalar master function, known as Hamilton function H. One of the features of Hamiltonian systems is that it conserves certain invariants in phase space, which constitute phase space as a *symplectic* space.

Whenever dissipation is negligible, most fundamental models of physics and mechanics are described by Hamiltonian systems. Hamiltonian systems have been the subject of numerous studies during the last two centuries in physics, mechanics, and astronomy, in problems ranging from the dynamics of elementary particles in accelerators to the dynamics of planetary objects in a space (Poincaré, 1892–99; Lichtenberg and Lieberman, 1992; MacKay and Meiss, 1987; Arnold et al., 1988).

Standard numerical methods of integrating systems of ordinary differential equations are not ideal for the purposes of solving Hamiltonian systems because the numerical approximation introduces non-Hamiltonian perturbations that completely change long-term behavior of the solutions. For this reason, special integration tools, known as symplectic integrators, have been developed for the numerical study of Hamiltonian systems (see, for example, reviews Sanz-Serna (1992); Sanz-Serna and Calvo (1993, 1994); Feng (1994)). The methods are constructed to preserve the symplectic properties of Hamiltonian systems by arranging each integration step to be a canonical transformation. Symplectic integration methods play an important role in the study of the long-term evolution of Hamiltonian systems.

Mappings are a powerful tool for studying Hamiltonian systems (see, e.g., Lichtenberg and Lieberman, 1992; MacKay and Meiss, 1987; Chirikov, 1979; Zaslavsky, 1985; Sagdeev et al., 1988; Zaslavsky et al., 1991). These maps are inherently constructed in symplectic from, and thereby preserve properties of Hamiltonian systems. This approach is most ideal to study the long-term evolution of a system, especially in cases where the system exhibits chaotic behavior caused by exponential divergency of orbits with close initial coordinates in phase space. Symplectic maps have been successfully employed in many problems of astronomy, plasma physics, fluid dynamics, accelerator physics, and others.

In spite of the extensive use of symplectic maps for many Hamiltonian problems during the last four decades, the derivation of generic symplectic maps from given Hamiltonian equations still remains somehow elusive. There are several approaches to construct symplectic maps from the continuous formulation of systems. One approach is based on the a priori assumption that the map has a symplectic form and the generating functions associated with the map are found from the equations of motion (Lichtenberg and Lieberman, 1992). Another method to construct symplectic maps is based on the assumption that a time-periodic perturbation acting on the integrable system may be replaced by periodic delta functions, which is equivalent to adding fast oscillating terms to the Hamiltonian (Wisdom, 1982; Zaslavsky, 1985; Sagdeev et al., 1988; Zaslavsky et al., 1991). Integration of the equations of motion along delta functions gives symplectic maps with the time-step equal to the period of perturbation. In particular, this method was used by Chirikov to derive the celebrated *standard map* (Chirikov, 1979; Lichtenberg and Lieberman, 1992). However, these methods have significant shortcomings and difficulties, and they do not have a good mathematical justification. Particularly, they do not establish more general forms of the maps, estimate their accuracy, and establish relations between variables of the original system and of the mapping.

Recently in Abdullaev (1999, 2002) a mathematically rigorous method to derive symplectic maps has been developed. Based on the Hamilton–Jacobi theory and the classical perturbation theory, it allows one to construct symplectic mappings for generic Hamiltonian systems in a rigorous and consistent way. It does not encounter the difficulties of more traditional methods.

The present book is devoted to the systematic theory of symplectic mappings for Hamiltonian systems and its application to different Hamiltonian problems. The method is based on the Hamilton–Jacobi method and perturbation theory of classical mechanics. This book compresses 13 chapters. The theory of construction of Hamiltonian maps is given in the first five chapters. Application of mapping methods to study physical problems described by Hamiltonian systems are given in Chaps. 6–13.

The first chapter contains the essential elements of Hamiltonian dynamics including the different formulations of Hamiltonian equations, constant of motion, the Hamilton–Jacobi method, and the formalism of action-angle variables. In the second chapter we have presented the methods of classical perturbation theory. Time-dependent perturbation theory that constitutes the basis for the construction of symplectic mappings has been also reiterated in this chapter. The current methods to construct the symplectic maps for generic Hamiltonian problems are discussed in the third chapter. The Hamilton–Jacobi method or the method of canonical transformation to construct Hamiltonian mappings is presented in the fourth chapter. There we also discussed the different forms of symplectic maps, their accuracy, and how they compare with standard numerical symplectic integration methods. Mappings near separatrix of Hamiltonian systems are constructed in Chap. 5 using canonical transformations of the variables. The construction of mappings near separatrix is illustrated in Chap. 6 for several Hamiltonian systems. In Chap. 7 we have applied the mapping methods to analyze some non-standard issues of Hamiltonian dynamics, namely, regular and chaotic dynamics in non-twist and non-smooth Hamiltonian systems. The rescaling invariance properties of Hamiltonian systems near the hyperbolic saddle points are discussed in Chap. 8. Chaotic transport in a stochastic layer and $\log \epsilon$-periodicity (ϵ is a perturbation amplitude) in $1\frac{1}{2}$-degrees of freedom Hamiltonian systems are studied in Chap. 9. Applications of symplectic mappings to the study of magnetic field lines in magnetically confinement devices are presented in the next three chapters. Particularly, in Chap. 10 the Hamiltonian formulation of magnetic field line equations in magnetically confinement devices, namely in tokamaks. Particularly, we discuss also the mapping methods to integrate magnetic field line equations, and mapping models of field lines in toroidal system. Chapters 11 and 12 are devoted to the application of mapping methods to study the magnetic structure in special devices of magnetic confinement, namely, in ergodic and poloidal divertors. In Chap. 13 other areas of physics, namely, wave propagation problems, accelerator physics and dynamical astronomy, where mapping methods play an important role, are discussed.

The book is intended for postgraduate students and researchers, physicists, and astronomers working in the areas of Hamiltonian dynamics and chaos, and its applications to plasma physics, hydrodynamics, celestial mechanics, dynamical astronomy, and accelerator physics. It should also be useful for applied mathematicians involved in analytical and numerical studies of dynamical systems. Readers are supposed to be familiar with the methods of classical mechanics on the level of Chaps. 1–3 and 7–9 of the book *Mathematical methods of classical mechanics* (Springer-Verlag, 1989) by V.I. Arnold.

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Sadrilla S. Abdullaev

Contents

1 Basics of Hamiltonian Mechanics

In this chapter we shall briefly recall the fundamental principles and methods of Hamiltonian mechanics which will be used throughout the book. This is for convenience of the reader and to fix notation. For more details, the reader might consult Arnold (1989). In particular, we shall give different formulations of Hamiltonian equations, and recall the invariants of motion. Special emphasis will be given to the Hamilton–Jacobi method and the action-angle formalism to integrate the equations of motion. These methods will be illustrated with the example of the pendulum. Finally, we shall shortly discuss modern methods of numerical symplectic integration of Hamiltonian systems.

1.1 Hamilton Equations

Consider a classical system with N degrees of freedom with q_i $(i = 1, \ldots, N)$ being the position coordinates of the particles of the system. In the classical (Newtonian) formulation the equations governing the time-evolution of the system are a set of second order ordinary differential equations for the positions q_i.

In the Hamiltonian formulation of classical mechanics the state of the system is characterized not only by its positions q_i, but also its momenta p_i, i.e., it is determined by coordinates in the so-called $2N$-dimensional *phase space* (q, p): N-coordinates $q = (q_1, \ldots, q_N)$ and $N-$ momenta $p = (p_1, \ldots, p_N)$. The time-evolution of the system is then governed by a set of $2N$ ordinary differential equations of first order in time t Hamilton (1834):

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\,, \qquad (i = 1, \ldots, N) \tag{1.1}$$

known as *Hamilton equations*, and determined by only one scalar master function $H = H(q, p, t)$ known as *Hamilton's function* (or *Hamiltonian*). The positions q_i and momenta p_i are called *canonical variables* and time t is an independent variable.

The Hamilton equations (1.1) with given initial conditions $q^{(0)} = (q_1^{(0)}, \ldots, q_N^{(0)})$ and $p^{(0)} = (p_1^{(0)}, \ldots, p_N^{(0)})$ at the moment $t = 0$ have unique solution

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$$q_i(t) = q_i(t, q_1^{(0)}, \dots, q_N^{(0)}, p_1^{(0)}, \dots, p_N^{(0)}) ,$$
$$p_i(t) = p_i(t, q_1^{(0)}, \dots, q_N^{(0)}, p_1^{(0)}, \dots, p_N^{(0)}) , \tag{1.2}$$

$(i = 1, \dots, N)$ at any arbitrary time instant $t > 0$ or $t < 0$.

Geometrically, the trajectories (1.2) may be considered as a *flow* of a $2N$-dimensional fluid in the phase space Lanczos (1962); Guckenheimer and Holmes (1983). The velocity field $\mathbf{v}$ of this fluid flow is $\mathbf{v} = (\dot{q}_1, \dots, \dot{q}_N, \dot{p}_1, \dots, \dot{p}_N)$. Below we shall see that this flow preserves some invariants of motion which are important in construction of mappings.

1.1.1 Invariants of Motion

Invariants (or *integrals*) of motion are most important to study the evolution of Hamiltonian systems. A function $F = F(q, p, t)$ is called an integral of motion if it does not change its initial value during the time evolution of the system. Using the Hamiltonian equations (1.1) it can be formally written as

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\} = 0 , \tag{1.3}$$

where the notation $\{F, \Phi\}$ stands for the Poisson bracket

$$\{F, \Phi\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial \Phi}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial \Phi}{\partial q_i} \right) . \tag{1.4}$$

The first integral of the system is the energy of a conservative system if the Hamiltonian H does not explicitly depend on time t, i.e., $H = H(q_1, \dots, q_N; p_1, \dots, p_N)$. It follows from (1.3) that $dH/dt = 0$ since $\partial H/\partial t = 0$ and $\{H, H\} \equiv 0$, and thus the energy of the conservative system is an integral of motion, $H = E = \text{const}$.

Another invariant property of Hamiltonian motion (or flow) comes from its similarity with a "incompressible fluid", i.e., an arbitrary volume of fluid element is unchanged during the motion. The condition of incompressibility for the phase fluid,

$$\nabla \cdot \mathbf{v} = \sum_{i=1}^{N} \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = 0 , \tag{1.5}$$

is satisfied for the canonical equations (1.1) with an arbitrary Hamiltonian $H = H(q, p, t)$, for conservative, as well as for non-conservative systems. This property of the Hamiltonian flow leads to conservation of any closed volume $\Omega(t)$ of phase space, i.e.,

$$V = \int_{\Omega(t)} dq_1 \dots dq_N dp_1 \dots dp_N = \text{const} . \tag{1.6}$$