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V. I. Arnol'd　S. P. Novikov (Eds.)

Dynamical Systems VII

Integrable Systems, Nonholonomic Dynamical Systems

动力系统 VII

可积系统，不完整动力系统

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

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Contents

I. Nonholonomic Dynamical Systems, Geometry of Distributions and Variational Problems

A.M. Vershik, V.Ya. Gershkovich

Translated from the Russian
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Introduction

0. A nonholonomic manifold is a smooth manifold equipped with a smooth distribution. This distribution is in general nonintegrable. The term 'holonomic' is due to Hertz and means 'universal', 'integral', 'integrable' (literally, $\acute{o}\lambda o\varsigma$ – entire, $\nu o\mu o\varsigma$ – law). 'Nonholonomic' is therefore a synonym of 'nonintegrable'.

A nonholonomic manifold is the geometric (or, more precisely, kinematic) counterpart of a nonholonomic dynamical system with linear constraints. As we shall see later, there are two main ways to construct dynamics on nonholonomic manifolds. They will be referred to (somewhat conventionally) as mechanical and variational, respectively.

The aim of the present survey is to give a possibly self-consistent account of the geometry of distributions and to lay down a foundation for a systematic study of nonholonomic dynamical systems. Our review is somewhat different in its character from other reviews in the present series. The reasons for this are rooted in the peculiar history of the subject as will be explained below.

Due to lack of space, the present volume includes only the first part of our survey which comprises geometry of distributions and variational dynamics. Geometry and dynamics of nonholonomic mechanical systems, nonholonomic connections, etc., will be described separately. However, in this introduction we will discuss all these topics, at least in historical aspect. Several problems of nonholonomic mechanics have already been considered in Volume 3 of this Encyclopaedia (Arnol'd, Kozlov and Nejshtadt [1985]). In this survey we give a modern exposition of some earlier results as well as new results which were not published previously.

1. Nonholonomic geometry and the theory of nonholonomic systems are the subject of numerous papers. A few of them are due to prominent geometers of the beginning of our century, others and still more numerous date back to the early past-war period. Nevertheless, this theory has not gained popularity in a broader mathematical audience. For instance, in most textbooks on Riemannian geometry and the calculus of variations there are hardly any facts on these subjects, save perhaps for the classical Frobenius theorem. Even the term 'nonholonomic' is scarcely mentioned. The exceptions are rather rare (e.g. the recent book of Griffiths [1983] where such results as the Chow-Rashewsky theorem are exposed).

There are many reasons to feel dissatisfied with this state of affairs. First of all, nonholonomic systems always held a sufficiently important place in mechanics. Mathematicians have always been thoroughly studying classical dynamics and the mathematical structures that are inherent to it. In view of this long-standing tradition the neglect of nonholonomic problems is almost striking. It is even not commonly known that nonholonomic mechanical problems cannot be stated as variational problems (cf. below).

Secondly, nonholonomic variational problems have much in common with optimal control problems which have been the subject of so many papers in recent years. This similarity has not been noticed until recently. Although the statements of nonholonomic variational problems are classical, some features of their solutions (e.g. the structure of the accessibility set, cf. Chapter 2) makes them quite close to non-classical problems. However, in textbooks on variational calculus even existence theorems for the simplest nonholonomic problems are lacking. The corresponding theorem in Chapter 2 of our survey is based on a recent observation made in connection with non-classical problems.

Thirdly, problems of thermodynamics (Gibbs, Carathéodory) and of quantum theory (Dirac) also lead to nonholonomic variational problems. A modern mathematical treatment of this kind of problems is still to be given.

Fourthly, nonholonomic problems are closely connected with the general theory of partial differential equatons. The best known results that display this connection are the theorems of L. Hörmander and A.D. Aleksandrov on hypoellipticity and hypoharmonicity. The study of these questions from a nonholonomic point of view has been actively pursued in recent years (we included some additional references to make this translation more up-to-date).

Finally, the general mathematical theory of dynamical systems (now in its maturity, as confirmed by the present edition) may well find in nonholonomic dynamics a vast source of new problems, examples, and paradoxes. We hope to support this view by the present survey.

2. We shall now give a brief review of the history of our subject. We believe that it will also explain the isolation of nonholonomic theory from the rest of mathematics which has continued up to the present time. A systematic development of the nonholonomic theory was started in the twenties and thirties, following a pattern which dates back to the turn of the century. This pattern was shaped in a series of classical papers on nonholonomic mechanics, due to many mathematicians and physicists: Hertz, Voss, Hölder, Chaplygin, Appell, Routh, Woronets, Korteweg, Carathéodory, Horac, Volterra, to mention a few. See Aleksandrov [1947], and Synge [1936] for a review. As mentioned by Grigoryan and Fradlin [1982], nonholonomic mechanical problems were already treated by Euler. However, it was not until the turn of the century that a clear understanding of their special features was gained. Hertz's name (Prinzipien der Mechanik, 1894) should be ranked first in this respect. A less known source of the theory comes from physics, namely, from the works of Gibbs and Carathéodory on the foundations of thermodynamics. They deal with contact structure which is the simplest example of nonholonomic structure. As for pure mathematics itself, in particular, the theory of distributions, which is an indispensable part of nonholonomic theories, we should begin with the theory of Pfaffian systems and subsequent works on the general theory of differential equations. The contribution of E. Cartan to this domain was of particular importance. He was the first to introduce differential forms and codistributions. Unfortunately, these tools were not widely used in nonholonomic problems.

Finally, we mention one trend of research in geometry going back to Issaly (1880), or, perhaps, even earlier. We mean the studies of nonholonomic surfaces, i.e. of nonholonomic 2-dimensional distributions in 3-dimensional space generalizing the ordinary theory of surfaces. This trend has been developed further by D.M. Sintsov and his school and by some other geometers in the Soviet Union. However, these works were not sufficiently known even to the experts and did not play a major role.

3. In the twenties when Levi-Civita and H. Weyl defined the notions of Riemannian and affine connections and discovered deep relations between mechanics and geometry, it became clear that nonholonomic mechanics should also serve as a source of new geometrical structures which, in turn, provide mechanics and physics with a convenient and concise language. This mutual interaction was started by Vranceanu and Synge. In the Soviet Union nonholonomic problems were activey advertized for by V. Kagan. In 1937 he proposed the following theme for the Lobachevsky prize competition (Vagner, [1940]): "To lay down the foundation of a general theory of nonholonomic manifolds. ⟨...⟩ Applications to mechanics, physics, or integration of Pfaffian systems are desirable".

The most important results on nonholonomic geometry and its connections with mechanics were obtained in the pre-war years and are due to Vranceanu and Synge, and also to Schouten and V. Vagner. In two short notes and an article a Romanian mathematician Vranceanu [1931] gave the first precise definition of a nonholonomic structure on a Riemannian manifold and outlined its relation to the dynamics of nonholonomic systems. Synge [1927, 1928] has studied the stability of the free motion of nonholonomic systems. In his work he anticipated the notion of the curvature of a nonholonomic manifold. It was formally introduced somewhat later and in two stages. First, Schouten defined what was later to be called truncated connection, i.e. parallel transport of certain vectors along certain vector fields. The geodesics of this connection are precisely the trajectories studied by Synge. Finally, a Soviet geometer V. Vagner made the next important step in a series of papers which won him the Lobachevsky prize of Kazan University for young Soviet mathematicians in 1937. He defined (in a very complicated way) the general curvature tensor which extends the Schouten tensor and satisfies all the natural conditions (e.g. it is zero if and only if the Schouten-Vranceanu connection is flat). In his subsequent papers Vagner extended and generalized his results. (Notably, Schouten was in the jury when Vagner defended his thesis.)[1]

4. The geometry of the straightest lines (i.e. classical mechanics of nonholonomic systems) is the subject of quite a few papers written mainly between the wars. By contrast, much less was done on the geometry of the shortest lines, i.e. on the variational theory of nonholonomic systems. The main

[1] Recently, a more modern exposition of Vagner's main work has appeared (Gorbatenko [1985].)

contributions bearing on this subject may be listed quite easily. The starting point for the theory was a paper of Carathéodory [1909] in which he proves that any two points on a contact manifold may be connected by an admissible curve. (This statement was already mentioned without proof in earlier papers, e.g. by Hertz.) It is interesting to notice that Carathéodory needed this theorem in connection with his work on foundations of thermodynamics, namely in order to justify the definition of thermodynamical entropy. Although this theorem has a kinematic nature, it may be used to define a variational, or nonholonomic, metric sometimes referred to as the Carnot-Carathéodory metric (see Chapter 3). An extension of this theorem to arbitrary totally nonholonomic manifolds was proved independently by Chow [1939] and by Rashevsky [1938]. Several results of the classical calculus of variations were extended to the nonholonomic case by Schoenberg who was specially studying variational problems. A comparison of mechanical and variational problems for nonholonomic manifolds was given by Franklin and Moore [1931]. An interpretation of nonholonomic variational problems in mechanical or optical terms has been proposed quite recently (Arnol'd, Kozlov and Nejshtadt [1985], Karapetyan [1981], Kozlov [1982a, b, 1983]).

5. Before we come to describe the contents of this paper it is worth commenting on the reasons which possibly account for the contrast between the importance of these subjects and their modest position in "main-stream" mathematics. The point is that most papers which bear on the subject were written extremely vaguely, even if one allows for the usual difficulties of "coordinate language". To put it in a better way, at that time it was practically impossible to give a clear exposition of the theory (which is far from being simple by any standards). The key concept that was badly needed was that of a connection on a principal bundle in its almost full generality. However, nothing of the kind was used at that time. The absence of an adequate language was really painful and resulted in enormous and completely unmanageable texts. It was difficult to extract from them even the main notions, to say nothing of theorems. As a consequence, these papers were not duly understood.[2] One needed such tools as jets, germs, groups of germs of diffeomorphisms, etc. This was already clear to E. Cartan. Although he never studied this subject specially, he frequently stresses that connections on principal fibre bundles should be used in nonholonomic problems (ironically, he quotes this idea in the same volume dedicated to the Lobachevsky prize competition that we already mentioned (Vagner [1940])).

All this may be the reason why the fundamental reshaping of differential geometry in a coordinate-free manner has left aside the nonholonomic theory. In the fifties and sixties this theory was already out of fashion and was to remain

[2] V. Vagner wrote in 1948: "The lack of rigour which is typical for differential geometry is reflected also in the absence of precise definitions of such notions as spaces, multi-dimensional surfaces, etc. Differential geometry is certainly dropping behind and this became even more dangerous when it lost its direct contacts with theoretical physics".

in obscurity for many years to come. Although many of the more recent papers on nonholonomic theory already used modern language, they were isolated from the fertilizing applications which had served as the starting point for geometers of the prewar time.

After a complete renewal of its language in the fifties and sixties modern differential geometry became one of the central parts of contemporary mathematics. Along with topology, the theory of Lie groups, the theory of singularities, etc., it has created a genuine mathematical foundation of mechanics and theoretical physics in general. An invariant formulation of dynamics permitted to apply various powerful tools in this domain. Gradually, this process has brought to bear on nonholonomic theory. Quite recently, the Schouten-Vranceanu connection was rediscovered by Vershik and Faddeev [1975]. (See also Godbillon [1969], Vershik [1984].) In this paper nonholonomic mechanics was exposed systematically in terms of differential geometry. In particular, it was shown that the local d'Alembert principle regarded as a precise geometric axiom implies the above mentioned theorem on geodesics in nonholonomic theory.

A good deal of the authors' efforts was aimed to extract geometrical ideas and constructions from the papers of the past years and to present them in a modern form. This goal has not been fully achieved, but it seems indispensable in order to develop systematically the qualitative and geometric theory of nonholonomic dynamical systems, in analogy with other theories of dynamical systems (e.g. Hamiltonian, smooth, ergodic, etc.). We tried to give the basic definitions and to describe the simplest (3-dimensional) examples. Many mathematicians and physicists have helped us by pointing out various scattered papers on the subject. We are particularly indebted to A.D. Aleksandrov, V.I. Arnol'd, A.M. Vassil'ev, A.M. Vinogradov, A.V. Nakhmann, V.V. Kozlov, N.V. Ivanov, Yu.G. Lumiste, Yu.I. Lyubich, N.N. Petrov, A.G. Chernyakov, V.N. Shcherbakov, and Ya.M. Eliashberg.

6. Let us now turn to a brief description of the general structure of the survey. Nonholonomic dynamics is based on the geometry of distributions which is the subject of Chapter 1. The simplest and best known example of a nonholonomic manifold is the contact structure, i.e. a maximally nonholonomic distribution of codimension 1. Since in the existing literature very little is available on distributions of larger codimension, we present the main definitions and the most important examples of distributions in Section 1 of Chapter 1. In Section 2 we study generic distributions and the classification problem. In particular, we present results on the existence of functional moduli of distributions for almost all growth vectors. We also briefly mention the notion of nilpotentization which is of particular importance, especially in the recent advances of the theory. (For more information consult the list of references which was enlarged to make this translation more up-to-date.) As already mentioned, there are two completely different dynamics associated with a nonholonomic Riemannian manifold: the dynamics of the 'straightests', or mechanical, and the dynamics of the 'shortests', or variational. The terms 'straightest' and 'shortest' were first intro-

duced in connection with mechanics by Hertz. The difference between them is briefly as follows. Using the distribution we may introduce the so called truncated connection (Schouten [1930]). The study of its geodesics and of the corresponding flow is connected with mechanics of systems with linear constraints, e.g., with the problem of rolling, etc. (The general theory comprises also nonlinear restrictions, cf. for instance Vershik and Faddeev [1975], Vershik and Gershkovich [1984].) These questions will be considered separately. If we restrict the metric to the distribution, we get a new metric on the manifold. Its geodesics (the shortests) are the subject of variational theory discussed in detail in Chapter 2. The phase space of a nonholonomic variational problem is the so called mixed bundle, i.e. the direct sum of the distribution regarded as a subbundle of the tangent bundle and of its annihilator regarded as a subbundle of the cotangent bundle. As already mentioned, variational problems also admit a proper mechanical interpretation.

In Section 1 of Chapter 2 we present the main notions and constructions related to nonholonomic variational problems, such as nonholonomic geodesic flow, nonholonomic metrics, the nonholonomic exponential mapping, wave fronts, etc. In Section 2 we compute the accessibility sets for nonholonomic problems (or control sets, in the language of control theory).

In Chapter 3 we consider nonholonomic variational problems on Lie groups and homogeneous spaces. As usual, problems on Lie groups offer the most important class of examples, as well as a training field to develop constructions and methods which may be then extended to the general setting. In Section 1 we discuss local questions: the wave front and the ε-sphere of a nonholonomic Riemannian metric. In Section 2 we present a complete description of the dynamics of systems associated with the nonholonomic geodesic flow on homogeneous spaces of 3-dimensional Lie groups. Our approach is based on the wide use of geometry (more precisely, nonholonomic Riemannian geometry) and of the theory of nilpotent Lie groups. These two sources provide a better understanding of various domains connected with the study of distributions, such as nonholonomic mechanics, the theory of hypoelliptic operators, etc. At the same time this approach leads to new problems in geometry and in the theory of Lie groups.

Chapter 1
Geometry of Distributions

§ 1. Distributions and Related Objects

1.1. Distributions and Differential Systems. In the sequel without further notice all objects, such as manifolds, functions, mappings, distributions, vector fields, forms, etc., are supposed to be infinitely differentiable.

Definition 1.1. Let X be a real smooth manifold without boundary, and let TX be its tangent bundle. A subbundle $V \subset TX$, i.e. a family $\{V_x\}_{x \in X}$ of linear subspaces $V_x \subset T_x$ of the tangent spaces which depend smoothly on the point $x \in X$, is called a *distribution* on X. If X is connected, the number $\dim V_x \equiv \dim V$ is called the *dimension* of the distribution.

In the simplest case a distribution has the following structure: there is a decomposition of X into submanifolds (leaves), and V_x is the tangent space to the leaf passing through x. In this case the distribution is said to be *integrable* and determines a foliation. Its leaves are called maximal integral submanifolds of the distribution; their dimension is equal to that of V. If $\dim V = 1$, V is always integrable and its integral submanifolds are (locally) integral curves of a vector field that generates V.

In this survey we shall be mainly interested in the opposite case of nonintegrable, or nonholonomic distributions. The simplest example of a nonholonomic distribution is provided by two-dimensional distributions in $\mathbb{R}^3$, for instance, given by $V_x = \mathrm{Lin}\{\partial/\partial x_1, -\partial/\partial x_2 + x_1\partial/\partial x_3\}$, $x = (x_1, x_2, x_3)$. As is frequently done, the distribution is defined here as the linear span of vector fields. Another way to define the same distribution is as follows: V is the null-space of a 1-form $x_1\, dx_2 + dx_3$ which defines a contact structure on $\mathbb{R}^3$. The description of a distribution as the set of null-spaces of a system of differential forms will also be frequently used.

A k-dimensional distribution on X may be regarded as a section of the *Grassmann bundle* associated with TX, i.e. of the bundle of k-dimensional subspaces of the tangent spaces. This construction equips the space $\mathscr{V}_k(X)$ of such distributions with the natural topology of C^∞-sections. If X is an open ball, then, for $k \geqslant 2$, nonintegrable distributions (and even maximally nonintegrable distributions, see Section 2) form an open dense subset in $\mathscr{V}_k(X)$. On the contrary, the integrability of a distribution, i.e. the existence of foliation, is an extremely rare (nowhere dense) event.

Definition 1.2. We shall say that a vector field ξ on X is *subordinate* (or *belongs*) *to* $V = \{V_x\}$ if $\xi_x \in V_x$ for all $x \in X$. If $V_x = \mathrm{Lin}\{\xi_x^i, i = 1, \ldots, n\}$, the vector fields ξ^i are said to *generate* V. An integral curve γ of a vector field belonging to V is called *admissible* (with respect to V): $\dot{\gamma}_x \in V_x$, $x \in X$.

Recall that the linear space (over $\mathbb{R}$) of smooth vector fields Vect X is a Lie algebra with respect to the Lie bracket $[\xi, \eta] = \xi\eta - \eta\xi$. (Vector fields may be regarded as derivations, and their product means composition of derivations.) Moreover, Vect X is a $C^\infty(X)$-module, since each $\xi \in$ Vect X may be multiplied by $f \in C^\infty(X)$.

If V is a distribution, the set of all vector fields belonging to V (i.e. the set of its sections) is a C^∞-submodule in Vect X. We introduce the following

Definition 1.3. A *differential system* on X is a linear space of vector fields on X which is a $C^\infty(X)$-module[1].

As we have explained above each distribution V gives rise to a differential system $N(V)$. However, there exist other differential systems that correspond to distributions with singularities, i.e. to fields of linear subspaces in TX of non-constant dimension. Such distributions appear quite naturally. For instance, the Lie bracket of two distributions may already have singularities, which motivates the necessity of the above definition.

Let F be a differential system. The set of all vectors $v \in T_xX$ for which there is a vector field $\xi \in F$ such that $\xi_x = v$ is a linear subspace $V_x \subset T_x$. If dim $V_x =$ const, F is generated by a distribution $\{V_x\} = V$, $F = N(V)$; otherwise such a distribution does not exist.

Proposition. *A differential system on a smooth manifold X is the space of sections of a distribution if and only if it is a projective $C^\infty(X)$-module.*

Recall that if X is an open ball, any projective module is free, hence, in local problems, differential systems that are free $C^\infty(X)$-modules are the same as distributions. (A free module is the direct sum of several copies of $C^\infty(X)$, a projective module is a direct summand of a free module).

Definition 1.4. A distribution V (a differential system N) is *involutive* if $N(V)$ (respectively, N) is a Lie algebra; in other words, the Lie bracket of two vector fields that are subordinate to V (respectively, belong to N) also belongs to V (respectively, to N).

In the sequel we shall maninly deal with local problems, and so it is useful to introduce local versions of the main definitions using the language of germs and jets. (Concerning the notions of germs, jets, etc, see Bröcker and Lander [1975], Golubitsky and Guillemin [1973].) Let W_n^r, $1 < r \leqslant \infty$, $n = 1, \ldots,$ be the space of *r-jets of vector fields* at $0 \in \mathbb{R}^n$, let ω_n be the space of germs of vector fields in a neighborhood of $0 \in \mathbb{R}^n$. The spaces $W_n^\infty \equiv W_n$ and ω_n are Lie algebras with respect to the Lie bracket of vector fields. Moreover, these spaces are modules over the ring of jets J_n^∞ and the ring of germs of functions E_n, respectively.

[1] Some authors use the term 'differential systems' for distributions. Since the latter term is well established in Russian literature we shall use the term 'differential system' for a different notion. Recall that the notion of $C^\infty(X)$-module means that vector fields from F may be multiplied by an arbitrary element of $C^\infty(X)$: $\forall \xi \in F\, \forall f \in C^\infty(X)\, f\xi \in F$.