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Walter Littman, editor

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David S. Jerison
Massachusetts Institute of Technology

Carlos E. Kenig
University of Minnesota, Minneapolis

Walter Littman
University of Minnesota, Minneapolis

Johannes C. C. Nitsche
University of Minnesota, Minneapolis

Steven Orey
University of Minnesota, Minneapolis

James Ralston
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STUDIES IN PARTIAL DIFFERENTIAL EQUATIONS

Walter Littman, editor
University of Minnesota, Minneapolis

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INTRODUCTION

This is a collection of articles about partial differential equations (p.d.e.) intended to be accessible to nonexperts. The articles vary somewhat in difficulty. A good background in analysis is recommended, and an elementary partial differential equations course would be helpful. The presence of a "local p.d.e. expert" to help one over an occasional hurdle would make for smoother sailing, of course.

The subject began more than two hundred years ago as a mathematical study of certain physical and geometric problems. *Laplace's equation*, for example, arose in fluid mechanics as well as in electrostatics (in the determination of the potential due to an electric charge distribution on a conducting surface). As their names indicate, the *wave equation* dealt with wave propagation problems arising in acoustics and optics, and the *heat equation* arose in the study of heat conduction problems. (For a short history of Laplace's equation and its role in the development of p.d.e., and analysis in general, see [5].)

For many years the study of these three equations constituted the bulk of research effort in p.d.e. As the more obvious questions began to be answered for these equations, mathematicians turned to generalization. For example, the natural question arose: "What classes of equations are the natural generalizations of the three basic ones, both with respect to the properties of their solutions, as well as with respect to the physical problems giving rise to them?" This was the origin of the three "types" of equations, *elliptic* (describing "steady state" phenomena), *hyperbolic* (describing wave-like phenomena), and *parabolic* (describing diffusion phenomena).

Until the late fifties the study of p.d.e. consisted, for all practical purposes, of the study of these three "types" of equations. At that time a new point of view emerged, the study of p.d.e. "independent of type," or, p.d.e. "of general type." The philosophy here is to ask, what is the relationship between the properties of the solutions of a (linear) equation and the nature of the coefficients? Very often there is an important intermediate step: "a priori inequalities." To be more precise, certain properties of the solutions are linked to certain inequalities that solutions of the equation or of a related equation must satisfy. These inequalities are in turn linked to formal relationships between the coefficients. Examples of such properties are smoothness, local existence, and the unique continuation property. Although this theory is now one of the mainstreams of p.d.e., perhaps the best single source is still Hörmander's book [6]. Of course we eagerly await that author's new book on the subject, which we understand is to appear in the near future.

Despite the emergence of the point of view just described, research in the area of equations of the traditional types has continued at an accelerated pace. In elliptic equations, many stubborn problems have been solved. For accounts of this success story see [15], [2]. For a neat exposition of higher order boundary problems see for example [1].

During all this activity, Laplace's equation has to a large extent been left out of the limelight. However, very recently there has been a renewed interest in this "grandfather equation" for problems in domains with a "rather rough" boundary. Our first article, by Carlos Kenig and David Jerison, gives an account of some of this recent research.

However, the minimal surface equation, the *nonlinear* elliptic equation *par excellence*, has been very much in the limelight during the last several decades, perhaps because of its unique position at the crossroads between geometry and p.d.e. Our second article, by

Johannes Nitsche, makes much of the story of this equation accessible to the nonspecialist.

One of the most dramatic mathematical developments has been the relationship discovered between parabolic equations and probability theory. This connection has become a two-way bridge enriching both disciplines. Our third article by Steven Orey gives an account of this relationship.

The study of propagation of singularities of solutions of equations (especially hyperbolic) has been assuming an ever-increasing role within the subject of p.d.e. Not only is it of physical interest (for example, it tells us how light rays travel) but it has many theoretical applications. The article by J. Ralston gives us a new, essentially self-contained, treatment of this difficult subject. For a more usual approach using heavier machinery see [13]. See also [9] for additional physical motivation and background material.

The article by Caffarelli and Littman gives an "elementary" derivation for a representation formula for solutions to $\Delta u - u = 0$ in R^n. It illustrates the use of Fourier series as well as generalized functions (more general than distributions) in p.d.e.

Let us point to a few other areas not covered so far before closing this introduction. First, there is the general area of singular integrals, pseudodifferential and Fourier integral operators, which have done so much to change the map of the subject. In addition to the well-known recent books by Treves and Taylor, we refer to [3], [4], [13] for a glimpse of the power that these tools have given to the subject.

For other problems in hyperbolic equations see [11], [12]; for scattering theory [8] and [16]; for variational inequalities [10] (first two chapters); for other articles of interest, see, for example, [7] and [14].

We hope that the articles in this collection will, at least to some extent, transfer the excitement of the research process from the researcher to the reader.

Finally, it is a pleasure to thank the individual contributors for the great effort they have put into this collection.

WALTER LITTMAN

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CONTENTS

BOUNDARY VALUE PROBLEMS ON LIPSCHITZ DOMAINS

David S. Jerison and Carlos E. Kenig**

INTRODUCTION

A harmonic function u is a twice continuously differentiable function on an open subset of $\mathbb{R}^n$, $n \geqslant 2$, satisfying the Laplace equation

$$\Delta u = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \right) u = 0.$$

Harmonic functions arise in many problems in mathematical physics. For example, the function measuring gravitational or electrical potential in free space is harmonic. A steady state temper-

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ature distribution in a homogeneous medium also satisfies the Laplace equation.

We will be concerned with the two basic boundary value problems for the Laplace equation, the Dirichlet and Neumann problems. Let D be a bounded, smooth domain in $\mathbb{R}^n$ and let f be a smooth (i.e., C^∞) function on ∂D, the boundary of D. The *Dirichlet problem* is to find (and describe) a function u that is harmonic in D, continuous in $\bar{D}$, and equals f on ∂D. This corresponds to the problem of finding the temperature inside a body D when one knows the temperature f on ∂D. The *Neumann problem* is to find a function u that is harmonic in D, belongs to $C^1(\bar{D})$, and satisfies $\partial u/\partial N = f$ on ∂D, where $\partial u/\partial N$ represents the normal derivative of u on ∂D. This corresponds to the problem of finding the temperature inside D when one knows the heat flow f through the boundary surface ∂D.

Our main purpose here is to describe results on the boundary behavior of u in the case of smooth domains and the extension of these results to the case of domains with corners (Lipschitz domains, Definition (1.27)). The boundaries of these domains have the borderline amount of smoothness necessary for the validity of theorems like the one stated below.

In a smooth domain, the method of layer potentials yields the existence of a solution u to the Dirichlet problem with boundary data $f \in C^{k,\alpha}(\partial D)$ and the bound

$$\|u\|_{C^{k,\alpha}(\bar{D})} \leqslant A_{k,\alpha}\|f\|_{C^{k,\alpha}(\partial D)}{}^{\dagger}, \qquad k = 0, 1, 2, \ldots .$$

$$0 < \alpha < 1$$

(Uniqueness of u follows immediately from the maximum principle.) In certain bad (nonsmooth) domains a solution for continuous boundary data f need not exist. The problem of describing these domains was settled completely by N. Wiener. (See Section 1.)

† The $C^{k,\alpha}$ norm is the supremum of all derivatives up to order k plus

$$\sup_{\substack{x, y \in \partial D \\ |\beta| = k}} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x - y|^\alpha}.$$

What happens if the size of f is measured in the L^2 norm? This is of interest as a measure of the variation in data even if we are only concerned with continuous functions: If $f_1 - f_2$ has small L^2 norm, we want to know that the corresponding solutions u_1 and u_2 are near each other. The wisdom of hindsight tells us that as long as we are going to examine all continuous functions in L^2 norm, it is no harder to consider arbitrary functions in L^2. Another reason to consider the L^2 norm is that it is better suited to the Neumann problem than C^k norms, even on smooth domains. Our first task is to formulate an appropriate theorem by examining a model case, namely the unit ball B. The first person to consider this sort of question was Fatou [19], who examined the case of the unit disc in $\mathbb{R}^2$ with $f \in L^\infty$ rather than L^2. In the first section we will prove the following theorem of Fatou type. Let $d\sigma$ denote surface measure of ∂B.

THEOREM. *Suppose that* $1 < p \leqslant \infty$ *and* $f \in L^p(\partial B, d\sigma)$. *Then there exists a unique harmonic function* u *in* B *such that* $\lim_{r \to 1} u(rQ) = f(Q)$ *for almost every* $Q \in \partial B$, *and*

$$\int_{\partial B} u^*(Q)^p \, d\sigma(Q) \leqslant C_p \int_{\partial B} |f(Q)|^p \, d\sigma(Q), \tag{$*$}$$

where $u^*(Q) = \sup_{0 \leqslant r < 1} |u(rQ)|$.

The theorem asserts that $f_r(Q) = u(rQ)$ converges to $f(Q)$ not only in L^p norm, but also in the sense of Lebesgue's dominated convergence. (In the analogous estimates to $(*)$ in the Neumann problem, u is replaced by the gradient of u. In that case the estimate fails for $p = \infty$, even if $\partial u / \partial N$ is continuous.)

There is an appropriate endpoint result at $p = 1$ with $L^1(\partial B, d\sigma)$ replaced by finite measures on ∂B. The result puts positive harmonic functions in B in one-to-one correspondence with positive measures on ∂B. In particular,

COROLLARY. *Every positive harmonic function* u *in* B *has a finite radial limit* $\lim_{r \to 1} u(rQ)$ *for almost every* $Q \in \partial B$.

In the theorem and its corollary, the radial limit can be replaced by a *nontangential limit*: if X tends to Q with $|X-Q| < (1+\alpha)\operatorname{dist}(X, \partial B)$ for some fixed $\alpha > 0$, then $u(X)$ has a limit for almost every Q.

The nature of the solution u to the Dirichlet problem changes as the domain becomes less smooth. This change is reflected in the need for alternative techniques to solve for u, but is best described in terms of a notion called harmonic measure. Let D be a bounded Lipschitz domain. As we shall see in the first section, Lipschitz domains are among the domains for which the solution to the Dirichlet problem exists for any $f \in C^0(\partial D)$. Given $X \in D$, the mapping $f \mapsto u(X)$ is a continuous linear functional on $C^0(\partial D)$. Therefore, by the Riesz representation theorem, there is a unique Borel measure ω^X on ∂D such that

$$u(X) = \int_{\partial D} f(Q)\, d\omega^X(Q).$$

ω^X is called *harmonic measure* for D evaluated at X. For example, harmonic measure for B evaluated at the origin is a constant multiple of surface measure: $\omega^0 = \sigma/\sigma(\partial B)$ (the mean value Theorem (1.6)).

Fix $X_0 \in D$, and denote $\omega = \omega^{X_0}$. The importance of harmonic measure to the boundary behavior of harmonic functions can be illustrated by the following theorem. If u is a positive harmonic function, then u has finite nontangential limits almost everywhere with respect to ω (see the corollary above, L. Carleson [9], and R. Hunt and R. Wheeden [24]). Conversely, given any set $E \subset \partial D$ with $\omega(E) = 0$, there is a positive harmonic function u in D with $\lim u(X) = \infty$ as $X \to Q$ for every $Q \in E$.

The difficulty with harmonic measure is that it is hard to calculate explicitly. In general, harmonic measure may be very different from surface measure. If D is a $C^{1,\alpha}$ domain (see (1.27)), then harmonic measure and surface measure are essentially identical in that each is a bounded multiple of the other. This can be proved by the classical method of layer potentials. Along the same lines, one can use layer potentials to solve the Dirichlet and Neumann problems with boundary data in L^p. On C^1 domains (see