Jan Mikusiński

THE BOCHNER INTEGRAL

BIRKHÄUSER

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by

JAN MIKUSIŃSKI

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MATHEMATISCHE REIHE

B

LEHRBÜCHER UND MONOGRAPHIEN
AUS DEM GEBIETE DER EXAKTEN WISSENSCHAFTEN

Preface

The theory of the Lebesgue integral is still considered as a difficult theory, no matter whether it is based the concept of measure or introduced by other methods. The primary aim of this book is to give an approach which would be as intelligible and lucid as possible. Our definition, produced in Chapter I, requires for its background only a little of the theory of absolutely convergent series so that it is understandable for students of the first undergraduate course. Nevertheless, it yields the Lebesgue integral in its full generality and, moreover, extends automatically to the Bochner integral (by replacing real coefficients of series by elements of a Banach space).

It seems that our approach is simple enough as to eliminate the less useful Riemann integration theory from regular mathematics courses.

Intuitively, the difference between various approaches to integration may be brought out by the following story on shoemakers.

A piece of leather, like in Figure 1, is given. The task consists in measuring its area. There are three shoemakers and each of them solves the task in his own way.

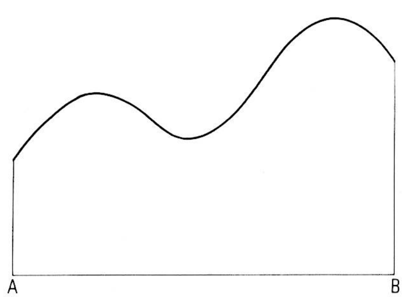

Fig. 1

The shoemaker R. divides the leather into a finite number of vertical strips and considers the strips approximately as rectangles. The sum of areas of all rectangles is taken for an approximate area of the leather (Figure 2). If he is not satisfied with the obtained exactitude, he repeats the whole procedure, by dividing the leather into thinner strips.

The shoemaker L. has another method. He first draws a finite number of horizontal lines. To each pair of adjacent lines he constructs a system of rectangles, as indicated in Figure 3. He finds the sum of areas of those rectangles, by multiplying their common height by the sum of lengths of their bases. He proceeds in the same way with each pair of adjacent lines

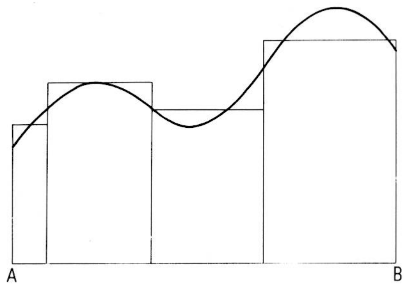

Fig. 2

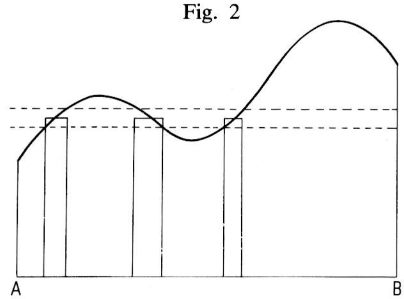

Fig. 3

and sums up the obtained results. If he is not satisfied with the obtained exactitude, he repeats the whole procedure with a denser set of horizontal lines.

The third shoemaker applies the following method. He takes a rectangle a_1 and considers its area as the first approximation. If he wants a more precise result, he corrects it by drawing further rectangles, as in Figure 4 or similarly. It is plain that, in case of Figure 4, the areas of rectangles a_1, a_2,

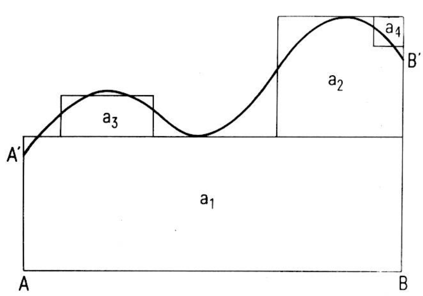

Fig. 4

a_3 are to be taken with positive signs, while the area of a_4 is to be taken with negative sign.

The reader acquainted with the theory of integration will easily recognize that the constructions shown in Figures 2 and 3 correspond to the Riemann and to the Lebesgue integrals, respectively. It is surprising that the

construction in Figure 4, which is so simple and natural, never was exploited in integration theory, before. This construction illustrates the idea of our definition in Chapter I, where the details are presented rigorously in the analytical language.
The main features of the theory are displayed in Chapters I–VII. We first select 3 basic properties **H**, **E**, **M**, and further properties of the integral are derived from them. Consequently, the theory applies not only to the Lebesgue and to the Bochner integrals, but also to each integral satisfying **H**, **E**, **M**, e.g., to the Daniell integral (see Chapter VIII, section 5).
Chapters VIII–XV contain some more special topics, selected after the taste of the author. Each chapter is preceded by a short information about its contents. There are also two appendices.

The galley proofs of the book were read by my friends Czesław Kliś, Krystyna Skórnik and my son Piotr Mikusiński. They introduced a number of improvements and corrections. It is a pleasure to express my thanks to them.

Jan Mikusiński.

Contents

Chapter I

The Lebesgue Integral

We define Lebesgue integrable functions as limits of series of brick functions (i.e., of characteristic functions of intervals) with a special type of convergence (section 3). This definition is equivalent to the original Lebesgue definition, but avoids mentioning measure or null sets. The integral of an integrable function is obtained, by definition, on integrating the corresponding series term by term. Although this definition is very simple, it requires a proof of consistency, for the function can expand in various series. That proof is preceded by two auxiliary theorems (Theorem 3.1 and Theorem 3.2) and makes the core of this chapter.

1. Step functions of one real variable

By a *brick* we shall mean a bounded half-closed interval $a \leqslant x < b$, where a and b are finite real numbers. A function whose values are 1 at the points of a brick J, and 0 at the points which do not belong to that interval, will be called a *brick function* and the brick J, its *carrier*. In other words, a brick function is the characteristic function of a brick, its carrier. By the *integral* $\int f$ of a brick function f we understand the length of its carrier: thus, if the carrier is $a \leqslant x < b$, then $\int f = b - a$.
By a *step function* f we mean a function which can be represented in the form

$$f = \lambda_1 f_1 + \cdots + \lambda_n f_n, \tag{1.1}$$

where $f_1, \ldots, f_n$ are brick functions and $\lambda_1, \ldots, \lambda_n$ are real coefficients. It is easily seen that the sum of two step functions is again a step function. Also the product of a step function by a real number is a step function. In other words, the set of all the step functions is a linear space. We assume as known the fact that, if necessary, we always can choose the brick functions $f_1, \ldots, f_n$ in the representation (1.1) so that their carriers are disjoint, i.e., have no common points. This implies in particular that the modulus (absolute value) $|f|$ of a step function is also a step function. By the integral $\int f$ of the step function (1.1) we mean

$$\int f = \lambda_1 \int f_1 + \cdots + \lambda_n \int f_n.$$

We assume as known the following facts. The value of the integral is independent of the representation (1.1). This means that, if we have another representation for the same function

$$f = \kappa_1 g_1 + \cdots + \kappa_p g_p,$$

then

$$\lambda_1\int f_1+\cdots+\lambda_n\int f_n=\kappa_1\int g_1+\cdots+\kappa_p\int g_p.$$

The integral has the following properties:

$$\int(f+g)=\int f+\int g,$$

$$\int(\lambda f)=\lambda\int f \quad (\lambda \text{ real number}),$$

$$f\leqslant g \quad \text{implies} \quad \int f\leqslant\int g.$$

In other words, the integral is a positive linear functional on the space of the step functions. Moreover

$$\left|\int f\right|\leqslant\int|f|.$$

We shall still prove

Theorem 1. *Given any step function f and a number $\varepsilon>0$, there is another step function g and a number $\eta>0$ such that*

$$g(x)-f(y)\geqslant 0 \quad \textit{for} \quad |x-y|<\eta,$$

$$\int g\leqslant\int f+\varepsilon.$$

Proof. Let $f=\lambda_1 f_1+\cdots+\lambda_n f_n$ and let $[a_i, b_i)$ $(i=1,\ldots,n)$ be the carrier of f_i. We assume that g_i is a brick function whose carrier is

$$\left[a_i-\frac{\varepsilon}{2n\lambda_i}, b_i+\frac{\varepsilon}{2n\lambda_i}\right).$$

This interval is greater than $[a_i, b_i)$, if $\lambda_i>0$, and smaller than $[a_i, b_i)$, if $\lambda_i<0$. If $a_i-\frac{\varepsilon}{2n\lambda_i}\geqslant b_i+\frac{\varepsilon}{2n\lambda_i}$ happens to hold for some i, then we put $g_i\equiv 0$. Letting $\eta=\min_i\left|\frac{\varepsilon}{2n\lambda_i}\right|$, we evidently have $\lambda_i g_i(x)-\lambda_i f_i(y)\geqslant 0$ for $|x-y|\leqslant\eta$ and $\lambda_i\int g_i\leqslant\lambda_i\int f_i+\frac{\varepsilon}{n}$. Hence the assertion follows for $g=\lambda_1 g_1+\cdots+\lambda_n g_n$.

2. Step functions of several real variables

The functions of q real variables $\xi_1,\ldots,\xi_q$ can be considered as functions of a point $x=(\xi_1,\ldots,\xi_q)$ in the q-dimensional space $\mathbf{R}^q$. By a *brick* J in $\mathbf{R}^q$ we shall mean the set of the points x such that $\xi_i\in J_i$, where $J_1,\ldots,J_q$

are one-dimensional bricks, as in section 1. In other words, J is the Cartesian product $J=J_1\times\cdots\times J_q$ of q one-dimensional bricks.
By *brick functions* we mean characteristic functions of bricks; the bricks are called the *carriers* of the corresponding functions. Thus, a brick function admits the value 1 on its carrier and vanishes outside it. By the *integral* $\int f$ of a function whose carrier is $J=J_1\times\cdots\times J_q$ we understand the product of the lengths of $J_1,\ldots,J_q$. Thus, if $f_1,\ldots,f_q$ are characteristic functions of $J_1,\ldots,J_q$, we can write

$$\int f=\int f_1\cdots\int f_q,$$

where the integrals on the right side have been defined in section 1.
Since our notation is the same in the case of several variables as in the case of one real variable, the definitions of a step function and of its integral can be repeated without any change. Also their properties are word for word the same, and Theorem 1 remains true (provided by $|x-y|$ we mean the distance between the points x and y). The proofs are essentially similar to those for a single real variable.

3. Lebesque integrable functions and their integrals

Given any real valued function f, defined in $\mathbf{R}^q$, we shall write

$$f\simeq\lambda_1f_1+\lambda_2f_2+\cdots, \tag{3.1}$$

where f_i are brick functions and λ_i are real numbers, if

1° $|\lambda_1|\int f_1+|\lambda_2|\int f_2+\cdots<\infty$, *and*

2° $f(x)=\lambda_1f_1(x)+\lambda_2f_2(x)+\cdots$ *at those points x at which the series converges absolutely.*

The functions satisfying (3.1) will be called *Lebesque integrable.* By the integral $\int f$ we understand the sum

$$\int f=\lambda_1\int f_1+\lambda_2\int f_2+\cdots.$$

We do not know, at first, whether $\int f$ is determined uniquely. However this will follow from the following basic theorem:

Theorem 3.1. *If 1° holds and, for every x, the series $\lambda_1f_1(x)+\lambda_2f_2(x)+\cdots$ either converges to a non-negative limit or diverges to $+\infty$, then $\lambda_1\int f_1+\lambda_2\int f_2+\cdots\geqslant 0$.*

Proof. Let

$$M=|\lambda_1|\int f_1+|\lambda_2|\int f_2+\cdots$$

and let ε be any fixed number with $0<\varepsilon<M$. Since M is finite, there is an index n_0 such that

$$\sum_{n_0+1}^{\infty} |\lambda_i| \int f_i < \varepsilon. \tag{3.2}$$

Let

$$g_n = \sum_{1}^{n_0} \lambda_i f_i + \sum_{n_0+1}^{n_0+n} |\lambda_i| \; f_i \qquad (n=1,2,\dots). \tag{3.3}$$

By Theorem 1, given any positive number ε, there are step functions h_n and positive numbers η_n such that

(i) $\quad h_n(x) \geq g_n(y) \quad$ for $\quad |x-y|<\eta_n$;

(ii) $\quad \int h_n \leq \int g_n + \varepsilon \cdot 2^{-n}$.

We evidently may assume that the sequence η_n is decreasing.
Let $k_n=(h_1-g_1)+\cdots+(h_n-g_n)+g_n$. Then $k_n(x) \geq h_n(x)$ and, by (i) and (ii), we have

(I) $\quad k_n(x) \geq g_n(y) \quad$ for $\quad |x-y|<\eta_n$;

(II) $\quad \int k_n \leq \int g_n + \varepsilon$;

(III) $\quad k_{n+1} \geq k_n$;

the last inequality follows because $k_{n+1}=k_n+(g_{n+1}-g_n)+(h_{n+1}-g_{n+1})$ and from the fact that the differences in the parentheses are non-negative. We shall show that, given any number $\delta>0$, we have

$$k_n \geq -\delta \quad \text{for sufficiently large } n. \tag{3.4}$$

In fact, suppose, conversely, that there is an increasing sequence of positive integers p_n and a sequence of points $x_{p_n} \in \mathbf{R}^q$ such that $k_{p_n}(x_{p_n})< -\delta$. It follows from (3.3) that the functions g_n are non-negative outside a fixed bounded interval (brick) J. Thus all the x_{p_n} must belong to J. Consequently, there is a subsequence t_n of p_n such that x_{t_n} converges to a limit y. Of course

$$k_{t_n}(x_{t_n})<-\delta \qquad (n=1,2,\dots). \tag{3.5}$$

On the other hand, there is an index $n_1>n_0$ such that $g_{n_1}(y)>-\delta$, for the sequence g_n converges, at every point x, to a positive limit or diverges to ∞. Hence by (I) we have $h_{n_1}(x)>-\delta$ for $|x-y|<\eta_{n_1}$. Since $x_{t_n}\to y$, there exists an index $r>n_1$ such that $|x_{t_r}-y|<\eta_{n_1}$. Consequently, we have by (III), $k_{t_r}(x_{t_r}) \geq k_{n_1}(x_{t_r})>-\delta$, which contradicts (3.5). This proves that (3.4) is true.
Since all the k_n are non-negative outside J, we may write instead of (3.4), $k_n \geq -\delta k$ for sufficiently large n, where k is the characteristic function of J.

The function k does not depend on δ. Thus, we can choose δ as small as to get the inequality $\int k_n \geqslant -\varepsilon$ for sufficiently large n. Hence and from (II) we obtain $\int g_n \geqslant -2\varepsilon$ for large n, i.e.,

$$\sum_1^{n_0} \lambda_i \int f_i + \sum_{n_0+1}^{n_0+n} |\lambda_i| \int f_i \geqslant -2\varepsilon$$

for sufficiently large n. Letting $n \to \infty$, we hence get

$$\sum_1^{n_0} \lambda_i \int f_i + \sum_{n_0+1}^{\infty} |\lambda_i| \int f_i \geqslant -2\varepsilon.$$

But $|\lambda_i| \leqslant \lambda_i + 2|\lambda_i|$, thus

$$\sum_1^{\infty} \lambda_i \int f_i + 2 \sum_{n_0+1}^{\infty} |\lambda_i| \int f_i \geqslant -2\varepsilon,$$

and by (3.2)

$$\sum_1^{\infty} \lambda_i \int f_i \geqslant -4\varepsilon.$$

Since ε may be chosen arbitrary small, we have $\sum_1^{\infty} \lambda_i \int f_i \geqslant 0$, which is the required inequality.

From Theorem 3.1 we obtain, as a corollary,

Theorem 3.2. *If f is integrable and $f \geqslant 0$, then $\int f \geqslant 0$.*

Proof. Let ε be any positive number and let (3.1) hold. There is an index n_0 such that (3.2) holds. At all the points where series (3.1) converges absolutely, we have

$$\sum_1^{n_0} \lambda_i f_i + \sum_{n_0+1}^{\infty} |\lambda_i| f_i \geqslant 0. \tag{3.6}$$

At the remaining points series (3.6) diverges to ∞. Thus, by Theorem 3.1,

$$\sum_1^{n_0} \lambda_i \int f_i + \sum_{n_0+1}^{\infty} |\lambda_i| \int f_i \geqslant 0. \tag{3.7}$$

Since $|\lambda_i| \leqslant \lambda_i + 2|\lambda_i|$, this implies

$$\sum_1^{\infty} \lambda_i \int f_i + 2 \sum_{n_0+1}^{\infty} |\lambda_i| \int f_i \geqslant 0. \tag{3.8}$$

Hence, in view of (3.2), we obtain $\sum_1^{\infty} \lambda_i \int f_i \geqslant -2\varepsilon$ and, by the definition of the integral, $\int f \geqslant -2\varepsilon$. Since ε is arbitrary, the inequality $\int f \geqslant 0$ follows.

From Theorem 3.2 it is easy to deduce the uniqueness of the integral $\int f$. Suppose that we have, besides (3.1), another expansion of the same function

$$f \simeq \kappa_1 g_1 + \kappa_2 g_2 + \cdots. \tag{3.9}$$