

Jan Mikusiński

THE BOCHNER INTEGRAL

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# THE BOCHNER INTEGRAL

by

JAN MIKUSIŃSKI



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The bochner integral


LEHRBÜCHER UND MONOGRAPHIEN

AUS DEM GEBIETE DER EXAKTEN WISSENSCHAFTEN

## Preface

The theory of the Lebesgue integral is still considered as a difficult theory, no matter whether it is based the concept of measure or introduced by other methods. The primary aim of this book is to give an approach which would be as intelligible and lucid as possible. Our definition, produced in Chapter I, requires for its background only a little of the theory of absolutely convergent series so that it is understandable for students of the first undergraduate course. Nevertheless, it yields the Lebesgue integral in its full generality and, moreover, extends automatically to the Bochner integral (by replacing real coefficients of series by elements of a Banach space).

It seems that our approach is simple enough as to eliminate the less useful Riemann integration theory from regular mathematics courses.

Intuitively, the difference between various approaches to integration may be brought out by the following story on shoemakers.

A piece of leather, like in Figure 1, is given. The task consists in measuring its area. There are three shoemakers and each of them solves the task in his own way.

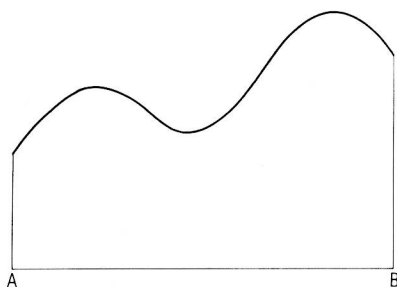


Fig. 1

The shoemaker R. divides the leather into a finite number of vertical strips and considers the strips approximately as rectangles. The sum of areas of all rectangles is taken for an approximate area of the leather (Figure 2). If he is not satisfied with the obtained exactitude, he repeats the whole procedure, by dividing the leather into thinner strips.

The shoemaker L. has another method. He first draws a finite number of horizontal lines. To each pair of adjacent lines he constructs a system of rectangles, as indicated in Figure 3. He finds the sum of areas of those rectangles, by multiplying their common height by the sum of lengths of their bases. He proceeds in the same way with each pair of adjacent lines

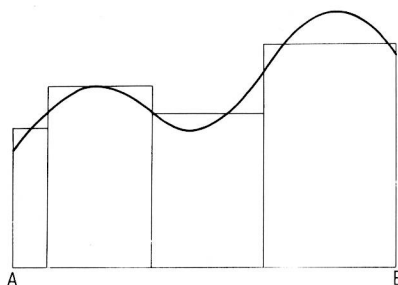


Fig. 2

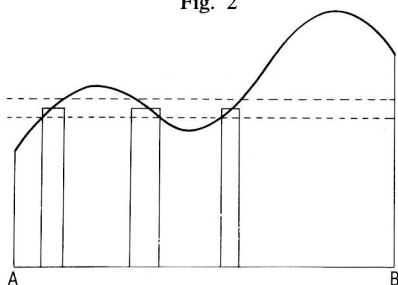


Fig. 3

and sums up the obtained results. If he is not satisfied with the obtained exactitude, he repeats the whole procedure with a denser set of horizontal lines.

The third shoemaker applies the following method. He takes a rectangle  $a_1$  and considers its area as the first approximation. If he wants a more precise result, he corrects it by drawing further rectangles, as in Figure 4 or similarly. It is plain that, in case of Figure 4, the areas of rectangles  $a_1$ ,  $a_2$ ,

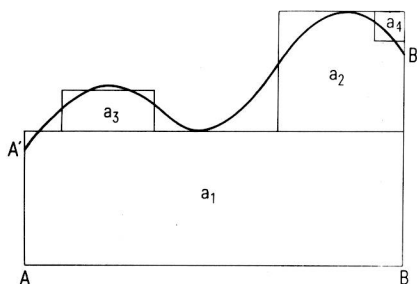


Fig. 4

$a_3$  are to be taken with positive signs, while the area of  $a_4$  is to be taken with negative sign.

The reader acquainted with the theory of integration will easily recognize that the constructions shown in Figures 2 and 3 correspond to the Riemann and to the Lebesgue integrals, respectively. It is surprising that the

construction in Figure 4, which is so simple and natural, never was exploited in integration theory, before. This construction illustrates the idea of our definition in Chapter I, where the details are presented rigorously in the analytical language.

The main features of the theory are displayed in Chapters I–VII. We first select 3 basic properties **H**, **E**, **M**, and further properties of the integral are derived from them. Consequently, the theory applies not only to the Lebesgue and to the Bochner integrals, but also to each integral satisfying **H**, **E**, **M**, e.g., to the Daniell integral (see Chapter VIII, section 5).

Chapters VIII–XV contain some more special topics, selected after the taste of the author. Each chapter is preceded by a short information about its contents. There are also two appendices.

The galley proofs of the book were read by my friends Czesław Kliś, Krystyna Skórnik and my son Piotr Mikusiński. They introduced a number of improvements and corrections. It is a pleasure to express my thanks to them.

Jan Mikusiński.



## Contents

<b>Chapter I: The Lebesgue Integral</b> . . . . .	1
1. Step functions of one real variable . . . . .	1
2. Step functions of several real variables . . . . .	2
3. Lebesgue integrable functions and their integrals . . . . .	3
<b>Chapter II: Banach Space</b> . . . . .	8
1. Vector space . . . . .	8
2. Normed space . . . . .	9
3. Convergence and absolute convergence . . . . .	9
4. Banach space . . . . .	12
<b>Chapter III: The Bochner Integral</b> . . . . .	15
1. Vector valued step functions . . . . .	15
2. Bochner integrable functions . . . . .	15
3. The modulus of a Bochner integrable function . . . . .	16
4. The Bochner integral . . . . .	20
5. Series of Bochner integrable functions . . . . .	21
<b>Chapter IV: Axiomatic Theory of the Integral</b> . . . . .	23
1. Elementary axioms . . . . .	23
2. Consequences of the elementary axioms . . . . .	24
3. Union and intersection of functions . . . . .	26
4. Null functions and equivalent functions . . . . .	26
5. The space $\tilde{U}$ . . . . .	27
6. Norm convergence . . . . .	28
7. Expansion axiom . . . . .	29
8. Convergence almost everywhere . . . . .	31
9. A few theorems referred to by name . . . . .	33
10. Asymptotic convergence . . . . .	35
<b>Chapter V: Applications to Set Theory</b> . . . . .	37
1. Characteristic functions of sets . . . . .	37
2. Convergence in measure . . . . .	39
<b>Chapter VI: Measurable Functions</b> . . . . .	40
1. Retracts of functions . . . . .	40
2. Measurable functions. The preserving axiom . . . . .	41
3. Elementary properties of measurable functions . . . . .	42
4. Measurable sets . . . . .	44
5. Integrals on arbitrary sets . . . . .	45

6. The Stone axiom . . . . .	47
7. Superposition of a continuous and a measurable function . . .	48
8. The product of measurable functions . . . . .	49
9. Local convergence in measure. Axiom Y. . . . .	50
<b>Chapter VII: Examples and Counterexamples . . . . .</b>	<b>53</b>
1. Examples of expansions of Lebesgue integrable functions . . .	53
2. Relations between the integration axioms . . . . .	56
3. Examples of non-measurable sets . . . . .	59
4. The outer and the inner cover of a set . . . . .	61
5. Norm convergence and pointwise convergence . . . . .	63
6. Local convergence in measure and asymptotic convergence . .	64
<b>Chapter VIII: The Upper Integral and Some Traditional Ap- proaches to Integration . . . . .</b>	<b>65</b>
1. The upper and the lower integrals . . . . .	65
2. Defining integrable functions by means of an upper integral . .	67
3. Further relations between the integral and the upper integral .	70
4. Riemann integrable functions . . . . .	70
5. The Daniell integral . . . . .	73
6. Integrals generated by $\sigma$ -measures . . . . .	78
<b>Chapter IX: Defining New Integrals by Given Ones . . . . .</b>	<b>83</b>
1. Restriction to a subset . . . . .	83
2. Hemmorphic integrals . . . . .	83
3. Substitution . . . . .	84
4. Multiplication of the integrand . . . . .	85
5. Sticking integrals together . . . . .	86
6. An example . . . . .	88
7. Defining <b>HEM</b> -integrals by <b>HEMS</b> -integrals . . . . .	89
<b>Chapter X: The Fubini Theorem . . . . .</b>	<b>91</b>
1. Cartesian products . . . . .	91
2. The Fubini theorem . . . . .	92
3. A generalization of the Fubini theorem . . . . .	94
4. The double integral as a <b>HEM</b> -integral . . . . .	96
5. Corollaries of the Fubini theorem . . . . .	99
6. A generalization of the Tonelli theorem . . . . .	102
<b>Chapter XI: Complements on Functions and Sets in the Euclidean q-Space . . . . .</b>	<b>106</b>
1. Locally integrable functions . . . . .	106
2. Local convergence . . . . .	106
3. Continuous functions . . . . .	107
4. Closed sets . . . . .	108
5. The distance of a point from a set . . . . .	109
6. Open sets . . . . .	109
7. The integral of a continuous function on a compact set . . .	112

<b>Chapter XII: Changing Variables in Integrals . . . . .</b>	<b>114</b>
1. Translation . . . . .	115
2. Linear substitution in one-dimensional case . . . . .	116
3. A particular transformation in $\mathbf{R}^q$ . . . . .	118
4. Linear transformation in $\mathbf{R}^q$ . . . . .	119
5. A decomposition theorem . . . . .	120
6. Measure of a linearly transformed set . . . . .	123
7. Anisotropic dilatation . . . . .	124
8. Further properties of linear transformations . . . . .	125
9. Invertible linear transformations . . . . .	127
10. Aureoles . . . . .	129
11. Antiaureoles . . . . .	131
12. Partial derivatives and full derivatives . . . . .	134
13. The one-dimensional case . . . . .	136
14. Basic properties of full derivatives . . . . .	137
15. Differential . . . . .	142
16. Invertibility of a fully differentiable function . . . . .	143
17. The Vitali covering theorem . . . . .	146
18. Jacobian theorem . . . . .	149
19. The substitution theorem . . . . .	158
20. Examples of application . . . . .	161
<b>Chapter XIII: Integration and Derivation . . . . .</b>	<b>164</b>
1. Two convergence theorems . . . . .	164
2. Primitive functions . . . . .	166
3. Local derivatives and local primitives . . . . .	169
4. A local convergence theorem . . . . .	173
5. Derivability almost everywhere . . . . .	178
6. Product of two local primitives . . . . .	180
7. Integration per parts . . . . .	181
<b>Chapter XIV: Convolution . . . . .</b>	<b>183</b>
1. Convolution of two functions . . . . .	183
2. Convolution of three functions . . . . .	186
3. Associativity of convolution . . . . .	188
4. Convolution dual sets of functions . . . . .	190
5. Application to vector valued functions . . . . .	193
6. Compatible sets . . . . .	195
7. Functions with compatible carriers . . . . .	196
8. Associativity of the convolution of functions with compatible carriers . . . . .	197
9. A particular case . . . . .	199
<b>Chapter XV: The Titchmarsh Theorem . . . . .</b>	<b>201</b>
1. Theorems on moments . . . . .	201
2. Formulation of the Titchmarsh theorem . . . . .	203
3. A lemma . . . . .	204

4. Proof of the Titchmarsh theorem . . . . .	207
5. Convex support of a convolution . . . . .	208
<b>Appendix I: Integrating Step Functions . . . . .</b>	<b>211</b>
1. Heaviside functions . . . . .	211
2. Bricks and brick functions . . . . .	212
3. Step functions . . . . .	213
4. Step functions of bounded carrier . . . . .	215
5. The integral of a step function of bounded carrier . . . . .	216
6. Fundamental properties of the integral . . . . .	218
<b>Appendix II: Equivalence of the New Definitions with the Old Ones . . . . .</b>	<b>222</b>
1. Equivalence of the original Lebesgue definition with the definition given in Chapter I . . . . .	222
2. Equivalence of the original Bochner definition with the definition given in Chapter II . . . . .	224
Exercises . . . . .	227
Bibliography . . . . .	232

# Chapter I

## The Lebesgue Integral

We define Lebesgue integrable functions as limits of series of brick functions (i.e., of characteristic functions of intervals) with a special type of convergence (section 3). This definition is equivalent to the original Lebesgue definition, but avoids mentioning measure or null sets. The integral of an integrable function is obtained, by definition, on integrating the corresponding series term by term. Although this definition is very simple, it requires a proof of consistency, for the function can expand in various series. That proof is preceded by two auxiliary theorems (Theorem 3.1 and Theorem 3.2) and makes the core of this chapter.

### 1. Step functions of one real variable

By a *brick* we shall mean a bounded half-closed interval  $a \leq x < b$ , where  $a$  and  $b$  are finite real numbers. A function whose values are 1 at the points of a brick  $J$ , and 0 at the points which do not belong to that interval, will be called a *brick function* and the brick  $J$ , its *carrier*. In other words, a brick function is the characteristic function of a brick, its carrier. By the *integral*  $\int f$  of a brick function  $f$  we understand the length of its carrier: thus, if the carrier is  $a \leq x < b$ , then  $\int f = b - a$ .

By a *step function*  $f$  we mean a function which can be represented in the form

$$f = \lambda_1 f_1 + \cdots + \lambda_n f_n, \quad (1.1)$$

where  $f_1, \dots, f_n$  are brick functions and  $\lambda_1, \dots, \lambda_n$  are real coefficients. It is easily seen that the sum of two step functions is again a step function. Also the product of a step function by a real number is a step function. In other words, the set of all the step functions is a linear space. We assume as known the fact that, if necessary, we always can choose the brick functions  $f_1, \dots, f_n$  in the representation (1.1) so that their carriers are disjoint, i.e., have no common points. This implies in particular that the modulus (absolute value)  $|f|$  of a step function is also a step function. By the integral  $\int f$  of the step function (1.1) we mean

$$\int f = \lambda_1 \int f_1 + \cdots + \lambda_n \int f_n.$$

We assume as known the following facts. The value of the integral is independent of the representation (1.1). This means that, if we have another representation for the same function

$$f = \kappa_1 g_1 + \cdots + \kappa_p g_p,$$

then

$$\lambda_1 \int f_1 + \cdots + \lambda_n \int f_n = \kappa_1 \int g_1 + \cdots + \kappa_p \int g_p.$$

The integral has the following properties:

$$\int (f + g) = \int f + \int g,$$

$$\int (\lambda f) = \lambda \int f \quad (\lambda \text{ real number}),$$

$$f \leq g \text{ implies } \int f \leq \int g.$$

In other words, the integral is a positive linear functional on the space of the step functions. Moreover

$$\left| \int f \right| \leq \int |f|.$$

We shall still prove

**Theorem 1.** *Given any step function  $f$  and a number  $\varepsilon > 0$ , there is another step function  $g$  and a number  $\eta > 0$  such that*

$$g(x) - f(y) \geq 0 \quad \text{for } |x - y| < \eta,$$

$$\int g \leq \int f + \varepsilon.$$

**Proof.** Let  $f = \lambda_1 f_1 + \cdots + \lambda_n f_n$  and let  $[a_i, b_i)$  ( $i = 1, \dots, n$ ) be the carrier of  $f_i$ . We assume that  $g_i$  is a brick function whose carrier is

$$\left[ a_i - \frac{\varepsilon}{2n\lambda_i}, b_i + \frac{\varepsilon}{2n\lambda_i} \right).$$

This interval is greater than  $[a_i, b_i)$ , if  $\lambda_i > 0$ , and smaller than  $[a_i, b_i)$ , if  $\lambda_i < 0$ . If  $a_i - \frac{\varepsilon}{2n\lambda_i} \geq b_i + \frac{\varepsilon}{2n\lambda_i}$  happens to hold for some  $i$ , then we put  $g_i \equiv 0$ . Letting  $\eta = \min_i \left| \frac{\varepsilon}{2n\lambda_i} \right|$ , we evidently have  $\lambda_i g_i(x) - \lambda_i f_i(y) \geq 0$  for  $|x - y| \leq \eta$  and  $\lambda_i \int g_i \leq \lambda_i \int f_i + \frac{\varepsilon}{n}$ . Hence the assertion follows for  $g = \lambda_1 g_1 + \cdots + \lambda_n g_n$ .

## 2. Step functions of several real variables

The functions of  $q$  real variables  $\xi_1, \dots, \xi_q$  can be considered as functions of a point  $x = (\xi_1, \dots, \xi_q)$  in the  $q$ -dimensional space  $\mathbf{R}^q$ . By a *brick*  $J$  in  $\mathbf{R}^q$  we shall mean the set of the points  $x$  such that  $\xi_i \in J_i$ , where  $J_1, \dots, J_q$

are one-dimensional bricks, as in section 1. In other words,  $J$  is the Cartesian product  $J = J_1 \times \cdots \times J_q$  of  $q$  one-dimensional bricks.

By *brick functions* we mean characteristic functions of bricks; the bricks are called the *carriers* of the corresponding functions. Thus, a brick function admits the value 1 on its carrier and vanishes outside it. By the *integral*  $\int f$  of a function whose carrier is  $J = J_1 \times \cdots \times J_q$  we understand the product of the lengths of  $J_1, \dots, J_q$ . Thus, if  $f_1, \dots, f_q$  are characteristic functions of  $J_1, \dots, J_q$ , we can write

$$\int f = \int f_1 \cdots \int f_q,$$

where the integrals on the right side have been defined in section 1.

Since our notation is the same in the case of several variables as in the case of one real variable, the definitions of a step function and of its integral can be repeated without any change. Also their properties are word for word the same, and Theorem 1 remains true (provided by  $|x - y|$  we mean the distance between the points  $x$  and  $y$ ). The proofs are essentially similar to those for a single real variable.

### 3. Lebesgue integrable functions and their integrals

Given any real valued function  $f$ , defined in  $\mathbf{R}^q$ , we shall write

$$f \approx \lambda_1 f_1 + \lambda_2 f_2 + \cdots, \quad (3.1)$$

where  $f_i$  are brick functions and  $\lambda_i$  are real numbers, if

- 1°  $|\lambda_1| \int f_1 + |\lambda_2| \int f_2 + \cdots < \infty$ , and
- 2°  $f(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \cdots$  at those points  $x$  at which the series converges absolutely.

The functions satisfying (3.1) will be called *Lebesgue integrable*. By the integral  $\int f$  we understand the sum

$$\int f = \lambda_1 \int f_1 + \lambda_2 \int f_2 + \cdots.$$

We do not know, at first, whether  $\int f$  is determined uniquely. However this will follow from the following basic theorem:

**Theorem 3.1.** *If 1° holds and, for every  $x$ , the series  $\lambda_1 f_1(x) + \lambda_2 f_2(x) + \cdots$  either converges to a non-negative limit or diverges to  $+\infty$ , then  $\lambda_1 \int f_1 + \lambda_2 \int f_2 + \cdots \geq 0$ .*

**Proof.** Let

$$M = |\lambda_1| \int f_1 + |\lambda_2| \int f_2 + \cdots$$

and let  $\varepsilon$  be any fixed number with  $0 < \varepsilon < M$ . Since  $M$  is finite, there is an index  $n_0$  such that

$$\sum_{n_0+1}^{\infty} |\lambda_i| \int f_i < \varepsilon. \quad (3.2)$$

Let

$$g_n = \sum_1^{n_0} \lambda_i f_i + \sum_{n_0+1}^{n_0+n} |\lambda_i| f_i \quad (n = 1, 2, \dots). \quad (3.3)$$

By Theorem 1, given any positive number  $\varepsilon$ , there are step functions  $h_n$  and positive numbers  $\eta_n$  such that

$$(i) \quad h_n(x) \geq g_n(y) \quad \text{for} \quad |x - y| < \eta_n;$$

$$(ii) \quad \int h_n \leq \int g_n + \varepsilon \cdot 2^{-n}.$$

We evidently may assume that the sequence  $\eta_n$  is decreasing.

Let  $k_n = (h_1 - g_1) + \dots + (h_n - g_n) + g_n$ . Then  $k_n(x) \geq h_n(x)$  and, by (i) and (ii), we have

$$(I) \quad k_n(x) \geq g_n(y) \quad \text{for} \quad |x - y| < \eta_n;$$

$$(II) \quad \int k_n \leq \int g_n + \varepsilon;$$

$$(III) \quad k_{n+1} \geq k_n;$$

the last inequality follows because  $k_{n+1} = k_n + (g_{n+1} - g_n) + (h_{n+1} - g_{n+1})$  and from the fact that the differences in the parentheses are non-negative. We shall show that, given any number  $\delta > 0$ , we have

$$k_n \geq -\delta \quad \text{for sufficiently large } n. \quad (3.4)$$

In fact, suppose, conversely, that there is an increasing sequence of positive integers  $p_n$  and a sequence of points  $x_{p_n} \in \mathbf{R}^q$  such that  $k_{p_n}(x_{p_n}) < -\delta$ . It follows from (3.3) that the functions  $g_n$  are non-negative outside a fixed bounded interval (brick)  $J$ . Thus all the  $x_{p_n}$  must belong to  $J$ . Consequently, there is a subsequence  $t_n$  of  $p_n$  such that  $x_{t_n}$  converges to a limit  $y$ . Of course

$$k_{t_n}(x_{t_n}) < -\delta \quad (n = 1, 2, \dots). \quad (3.5)$$

On the other hand, there is an index  $n_1 > n_0$  such that  $g_{n_1}(y) > -\delta$ , for the sequence  $g_n$  converges, at every point  $x$ , to a positive limit or diverges to  $\infty$ . Hence by (I) we have  $h_{n_1}(x) > -\delta$  for  $|x - y| < \eta_{n_1}$ . Since  $x_{t_n} \rightarrow y$ , there exists an index  $r > n_1$  such that  $|x_{t_r} - y| < \eta_{n_1}$ . Consequently, we have by (III),  $k_{t_r}(x_{t_r}) \geq k_{n_1}(x_{t_r}) > -\delta$ , which contradicts (3.5). This proves that (3.4) is true.

Since all the  $k_n$  are non-negative outside  $J$ , we may write instead of (3.4),  $k_n \geq -\delta k$  for sufficiently large  $n$ , where  $k$  is the characteristic function of  $J$ .



The function  $k$  does not depend on  $\delta$ . Thus, we can choose  $\delta$  as small as to get the inequality  $\int k_n \geq -\varepsilon$  for sufficiently large  $n$ . Hence and from (II) we obtain  $\int g_n \geq -2\varepsilon$  for large  $n$ , i.e.,

$$\sum_1^{n_0} \lambda_i \int f_i + \sum_{n_0+1}^{n_0+n} |\lambda_i| \int f_i \geq -2\varepsilon$$

for sufficiently large  $n$ . Letting  $n \rightarrow \infty$ , we hence get

$$\sum_1^{n_0} \lambda_i \int f_i + \sum_{n_0+1}^{\infty} |\lambda_i| \int f_i \geq -2\varepsilon.$$

But  $|\lambda_i| \leq \lambda_i + 2|\lambda_i|$ , thus

$$\sum_1^{\infty} \lambda_i \int f_i + 2 \sum_{n_0+1}^{\infty} |\lambda_i| \int f_i \geq -2\varepsilon,$$

and by (3.2)

$$\sum_1^{\infty} \lambda_i \int f_i \geq -4\varepsilon.$$

Since  $\varepsilon$  may be chosen arbitrary small, we have  $\sum_1^{\infty} \lambda_i \int f_i \geq 0$ , which is the required inequality.

From Theorem 3.1 we obtain, as a corollary,

**Theorem 3.2.** *If  $f$  is integrable and  $f \geq 0$ , then  $\int f \geq 0$ .*

**Proof.** Let  $\varepsilon$  be any positive number and let (3.1) hold. There is an index  $n_0$  such that (3.2) holds. At all the points where series (3.1) converges absolutely, we have

$$\sum_1^{n_0} \lambda_i f_i + \sum_{n_0+1}^{\infty} |\lambda_i| f_i \geq 0. \quad (3.6)$$

At the remaining points series (3.6) diverges to  $\infty$ . Thus, by Theorem 3.1,

$$\sum_1^{n_0} \lambda_i \int f_i + \sum_{n_0+1}^{\infty} |\lambda_i| \int f_i \geq 0. \quad (3.7)$$

Since  $|\lambda_i| \leq \lambda_i + 2|\lambda_i|$ , this implies

$$\sum_1^{\infty} \lambda_i \int f_i + 2 \sum_{n_0+1}^{\infty} |\lambda_i| \int f_i \geq 0. \quad (3.8)$$

Hence, in view of (3.2), we obtain  $\sum_1^{\infty} \lambda_i \int f_i \geq -2\varepsilon$  and, by the definition of the integral,  $\int f \geq -2\varepsilon$ . Since  $\varepsilon$  is arbitrary, the inequality  $\int f \geq 0$  follows.

From Theorem 3.2 it is easy to deduce the uniqueness of the integral  $\int f$ . Suppose that we have, besides (3.1), another expansion of the same function

$$f \approx \kappa_1 g_1 + \kappa_2 g_2 + \cdots. \quad (3.9)$$