Mathematical Topics
for Engineering
and Science Students

Ordinary Differential Equations

L B Jones

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Preface

The material presented in this book is based on lecture courses delivered by myself to engineering and science students over a period of years. It is hoped that the material presented will provide the reader with a good introductory knowledge of the techniques of analytic and numerical solution of ordinary differential equations. Any student wishing to pursue any part of the subject in further depth can refer to books listed in the short bibliography at the end of this book. Most of the illustrative examples have physical applications and are drawn from various fields. The problems which are given for solution by the student are however given in mathematical form, so that they are of general application rather than being restricted to any one branch of engineering or science. Many problems are given, and answers to all are listed at the end of the book.

In this book when, for example, integrals need to be evaluated reference is made to particular formulae in Barnett and Cronin (1975).

I should like to thank Dr S. Barnett for his many helpful suggestions during the preparation of the manuscript, and for invaluable help in correcting the proofs. Also to Dr J. A. Grant and Mr G. Eccles for their helpful suggestions, to Mrs M. B. Balmforth, Mrs J. Foster and Miss V. M. Morton for typing the manuscript, to Mrs J. Braithwaite and Mr S. Teal for preparing the drawings, and the University of Bradford for permission to make use of examples from University examination papers in preparing the examples and problems of this book.

L. B. Jones
September, 1975

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CHAPTER ONE

Introductory Ideas

1.1 Introduction

A differential equation is an equation that involves derivatives. The following are examples of differential equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \tag{1.1}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = \sin x \tag{1.2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \sin y = 0 \tag{1.3}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{1.4}$$

Equations (1.1)–(1.3) involve only ordinary derivatives, and they are called *ordinary differential equations.* Equation (1.4) involves partial derivatives and so is called a *partial differential equation.*

As the title of the book states, we will be concerned only with ordinary differential equations. Throughout this book, derivatives will be denoted in various ways. If y is a function of x, and assuming that the function is differentiable to whatever order is required, then the first derivative will be denoted by any of the following forms:

$$\frac{\mathrm{d}y}{\mathrm{d}x}, y', y^{(1)}$$

the second derivative by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}, y'', y^{(2)}$$

and the nth derivative by

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}, y^{(n)}$$

Equation (1.3) can therefore be expressed in the following alternative forms:

$$y'' + (y')^2 + \sin y = 0$$

$$y^{(2)} + [y^{(1)}]^2 + \sin y = 0$$

Given a differential equation involving y, x and the derivatives $\mathrm{d}^n y/\mathrm{d}x^n$, we wish to determine the resulting dependence of y on x. That is, we wish to find a relationship between y and x, say of the form $y = f(x)$, such that the relationship satisfies the differential equation. Such a relationship is called *a solution* of the differential equation.

For example, by straightforward differentiation it is easily seen that

$$y = \tfrac{1}{2}x^2, y = \tfrac{1}{2}x^2 - 1, y = \tfrac{1}{2}x^2 + 2$$

are all solutions to the differential equation (1.1), while

$$y = x, y = \tfrac{1}{3}x^2, y = \tfrac{1}{3}x^3 + 1$$

are not solutions.

Solutions to differential equations must often satisfy certain additional conditions. For example, the path of a projectile may be required subject to its passing through a given point in space with a given velocity at that point; or the deflection of a beam under load may be required subject to its being clamped at two or more places. If all the conditions are prescribed at a given point, as for the projectile, they are called *initial conditions*. If they are prescribed at different points, as for the beam, they are called *boundary conditions*.

Solutions may be obtained in closed form involving known functions, when they are called *analytic solutions*, or they may be obtained as *approximate solutions* which are for practical purposes close enough to the exact analytic solution over a specified range of values; or they may be obtained as *numerical solutions*, usually using a computer. In some cases, however, it may be sufficient to determine only the main characteristics of the solution, rather than the solution itself.

Example 1.1 Consider the differential equation

$$\frac{dx}{dt} = \tfrac{1}{2}(1 - x^2) \qquad \text{subject to} \qquad x = 0 \text{ when } t = 0 \tag{1.5}$$

x can be taken as the displacement of a particle and t as time.

(a) The analytic solution, which will be obtained in Problem 2.2(a), is

$$x(t) = \frac{e^t - 1}{e^t + 1} \tag{1.6}$$

This can be checked by direct substitution into the differential equation, and can also be seen to satisfy the given condition.

(b) An approximate solution valid for small values of time is

$$x = \tfrac{1}{2}t - \tfrac{1}{24}t^3$$

which is obtained by solution in series as described in Chapter 6. Terms involving t^n where $n \geqslant 5$ have been neglected, and this gives a measure of the accuracy of the approximation.

(c) Using a fourth-order Runge–Kutta method, as described in Section 3.3.2, the solution is

t	0	0·1	0·2	0·3
x	0	0·0500	0·0997	0·1489

where the values of x as given by this numerical method are accurate to about $\pm 10^{-4}$.

(d) The main characteristics of the motion given by (1.5) can be obtained by noting that when $dx/dt > 0$, x increases as t increases, and that when $dx/dt < 0$, x decreases when t increases. Hence for the differential equation (1.5), x increases with time when $-1 < x < 1$ and x decreases with time when $x > 1$ and $x < -1$. When $x = \pm 1$ the velocity of the particle is zero so that the points $x = \pm 1$ are equilibrium points, that is points at which the particle can remain at rest for all values of time. This information is illustrated in Fig. 1.1, the arrows denoting the motion as time increases, starting at various initial points. It will be noted that $x = 1$ is a stable point in that any particle on either side of $x = 1$ approaches $x = 1$, while $x = -1$ is an unstable point.

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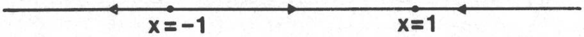

Fig. 1.1

Returning to the given condition $x = 0$ when $t = 0$, we see from Fig. 1.1 that x will increase with time and eventually approach $x = 1$. This is confirmed by the analytic solution, since from (1.6)

$$\lim_{t \to \infty} x(t) = 1 \qquad \text{and} \qquad 0 \leqslant x(t) < 1, \quad t \geqslant 0$$

It will be shown in the next section how differential equations arise, and that in most physical situations the differential equation which models a situation is based on a number of approximating assumptions. Accurate solutions of the differential equations are needed when the question of the validity of the model arises, since any discrepancy between the theoretical and experimental results needs to be attributed to the model and not to its solution. When the model is known to be a good representation of the physical situation, then the scientist or engineer may from practical considerations only require the solution to be accurate within prescribed limits. If an analytic solution cannot be found, then any numerical or approximate solutions need to have an accuracy within those prescribed limits. It is therefore necessary in any numerical or approximate solution to be able to quote the order of the error.

1.2 Formulation of differential equations

Differential equations arise in many ways, but we shall mainly be interested in those that result from the mathematical representation of physical situations.

Let us consider in a little detail the mathematical representation of the path of a projectile. The horizontal and vertical axes are denoted by x and y respectively (Fig. 1.2). The projectile P has mass m and is projected with speed V at an angle α to the horizontal.

The only force acting on the projectile is the constant force of gravity, mg, acting in a vertical (downward) direction. Velocity is the rate of change of distance travelled, so that the components of the velocity of the projectile in the x and y directions are $\mathrm{d}x/\mathrm{d}t$ and

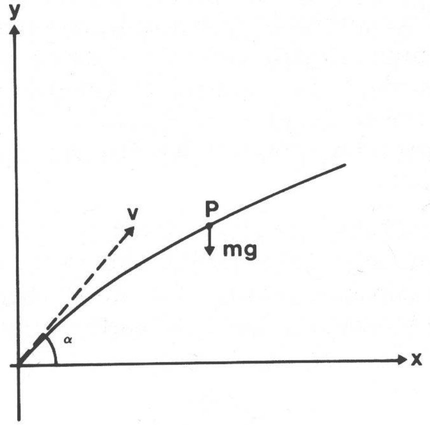

Fig. 1.2

dy/dt respectively. Similarly the components of the acceleration in the x and y directions are d^2x/dt^2 and d^2y/dt^2 respectively.

The differential equations representing the path of the projectile are now obtained by using Newton's law of motion, which states that force equals mass multiplied by acceleration.

Hence in the x direction,

$$m\frac{d^2x}{dt^2} = 0 \tag{1.7}$$

and in the y direction

$$m\frac{d^2y}{dt^2} = -mg$$

The above pair of differential equations together with the initial conditions

$$\left.\begin{array}{cc} & x = y = 0 \\ dx/dt = V\cos\alpha & dy/dt = V\sin\alpha \end{array}\right\} t = 0$$

provide a mathematical model of the motion. The solution of the differential equations with the initial conditions results in a relation between y and x which is the equation for the path of the projectile. This equation is obtained later in Problem 4.9, and is the equation of a parabola, as is well known. This parabolic path is the exact solution of the differential equations, but we now question whether this path coincides with or approximates closely the actual path of the projectile. In other words, is the mathematical model an accurate enough representation of the actual physical situation? We look at some of the assumptions or approximations that have been made to obtain our mathematical model.

(a) We have assumed that the gravitational force is constant. In fact the force is a function of its position above the earth. Taking the earth to be of constant density of radius R, then the gravitational acceleration g at height h above the earth is more closely given by

$$g = \frac{g_0 R^2}{(h + R)^2}$$

where g_0 is the gravitational acceleration at the earth's surface. When $h \ll R$, the value of g can be approximated by g_0, so that when we are concerned with the trajectory of a bullet, golf ball or even flight of an aircraft, the gravitational force can be approximated by a constant value. When we are dealing with the motion of a satellite, the gravitational force has to be taken as obeying an inverse square law, as above. Assuming that the gravitational force due to other bodies such as the sun is small compared with that of the earth, the motion is an orbit with the centre of the earth as focus. Strictly, even when dealing with motion near the earth's surface the path is an orbit with the earth's centre as focus, but this orbit over the small range of distances involved is very closely approximated by the parabolic path obtained on taking a constant gravitational force.

(b) We have assumed that the only force acting on the projectile is that due to gravitation. It is very likely, however, that other forces act on the projectile and that they are not negligible in comparison with the gravitational force. For instance, if we consider motion through air, there is a force resisting motion due to friction between the air and the projectile, and there might be other forces depending upon the shape of the projectile: for example, wings are designed to provide a lifting force (force in the vertical direction). These forces for a particular projectile have to be obtained from aerodynamic theory.

We now consider the case in which the resultant of the forces, other than gravitational, acting on the projectile opposes its motion and is proportional to its velocity (Fig. 1.3). Let v be the speed of the projectile and θ be the angle that the tangent to the path makes with the horizontal.

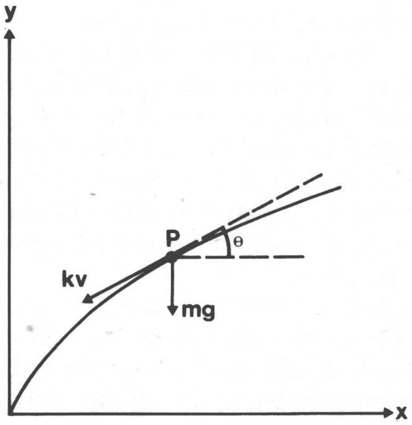

Fig. 1.3

The component of force in the x direction is $-kv\cos\theta$, and in the y direction is $-kv\sin\theta$. But $v\cos\theta$ is the component of the velocity in the direction of the x axis, and hence $v\cos\theta = \mathrm{d}x/\mathrm{d}t$, and similarly $v\sin\theta = \mathrm{d}y/\mathrm{d}t$.

The differential equations giving the motion are now

$$\begin{aligned} m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -k\frac{\mathrm{d}x}{\mathrm{d}t} \\ m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -k\frac{\mathrm{d}y}{\mathrm{d}t} - mg \end{aligned} \qquad (1.8)$$

with the same initial conditions as taken previously. Equations (1.8) are the model needed when the forces are as stated. However, the air forces acting on the projectile need not lie in the x,y plane, and if they do not, the path of the projectile is not two dimensional. A set of three differential equations is needed to represent the projectile path. An example of this is well known to all golfers when a certain spin imparted to a golf ball results in a slice or hook shot.

We see from the above discussion that our original set of differential equations is an adequate approximation to the physical situation, provided that the projectile stays close to the earth's surface and that air or other forces are of very small magnitude in comparison to the gravitational force. If any of these conditions are not satisfied, then a different set of differential equations is needed.

It must therefore be kept in mind that representation of the physical situation by a set of differential equations is only an approximation, albeit a valid approximation, to the real case. This arises because the basic physical 'law' may be valid only over a certain domain of the variables, and because expressions for quantities, such as forces, may also be valid only over a certain domain. However, within the assumptions made, the model is an exact representation of an idealised physical situation.

A few examples follow. The 'laws' used are quoted and the mathematical model obtained. One assumption in each example is commented upon, and the student should think about any other assumptions that have been made in setting up the model, and also about the validity of the 'laws' quoted, consulting, when necessary, books with the required physical background.

1.2.1 Cooling of a hot body

A body initially at temperature T_0 is surrounded by a medium at constant temperature T_m ($T_0 > T_\mathrm{m}$). The cooling of the body is taken to be governed by Newton's law of cooling, which states that the rate of decrease of the body temperature is proportional to the difference between the body temperature and that of the surrounding medium.

Let T be the temperature of the body at time t.

Rate of decrease of body temperature $= -\,\mathrm{d}T/\mathrm{d}t$.

The minus sign occurs since $\mathrm{d}T/\mathrm{d}t$ is the rate of increase of temperature with respect to time.

The excess temperature = $T - T_m$.
Hence

$$-\frac{dT}{dt} = k(T - T_m) \tag{1.9}$$

where k (> 0) is the constant of proportionality.

This is the differential equation governing the cooling of the body and is subject to the *initial condition*

$$T = T_0 \qquad \text{when } t = 0$$

(At any time t, all points of the body are assumed to have the same temperature T.)

1.2.2 A two-mass-spring system

The system is hanging vertically as shown in Fig. 1.4. The two similar weightless springs have natural length L and modulus of elasticity λ.

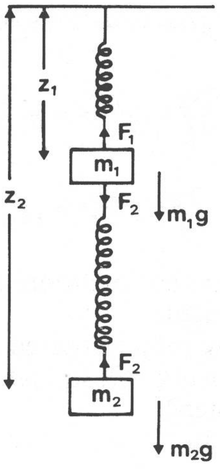

Fig. 1.4

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