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A. N. Parshin　I. R. Shafarevich (Eds.)

Number Theory IV
Transcendental Numbers

数 论 IV
超越数

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

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Preface

This book was written over a period of more than six years. Several months after we finished our work, N. I. Fel'dman (the senior author of the book) died. All additions and corrections entered after his death were made by his coauthor.

The assistance of many of our colleagues was invaluable during the writing of the book. They examined parts of the manuscript and suggested many improvements, made useful comments and corrected many errors. I much have pleasure in acknowledging our great indebtedness to them.

Special thanks are due to A. B. Shidlovskii, V. G. Chirskii, A. I. Galochkin and O. N. Vasilenko. I would like to express my gratitude to D. Bertrand and J. Wolfart for their help in the final stages of this book.

Finally, I wish to thank Neal Koblitz for having translated this text into English.

August 1997 *Yu. V. Nesterenko*

Transcendental Numbers

N. I. Fel'dman and Yu. V. Nesterenko

Translated from the Russian
by Neal Koblitz

Contents

Notation 9

Introduction 11

0.1 Preliminary Remarks 11
0.2 Irrationality of $\sqrt{2}$ 11
0.3 The Number π 13
0.4 Transcendental Numbers 14
0.5 Approximation of Algebraic Numbers 15
0.6 Transcendence Questions and Other Branches of Number Theory 16
0.7 The Basic Problems Studied in Transcendental Number Theory 17
0.8 Different Ways of Giving the Numbers 19
0.9 Methods 20

Chapter 1. Approximation of Algebraic Numbers 22

§1. Preliminaries 22
1.1. Parameters for Algebraic Numbers and Polynomials 22
1.2. Statement of the Problem 22
1.3. Approximation of Rational Numbers 23
1.4. Continued Fractions 24
1.5. Quadratic Irrationalities 25
1.6. Liouville's Theorem and Liouville Numbers 26

1.7. Generalization of Liouville's Theorem 27
§2. Approximations of Algebraic Numbers and Thue's Equation 28
2.1. Thue's Equation 28
2.2. The Case $n = 2$ 30
2.3. The Case $n \geq 3$ 30
§3. Strengthening Liouville's Theorem. First Version of Thue's Method 31
3.1. A Way to Bound $q\theta - p$ 31
3.2. Construction of Rational Approximations for $\sqrt[n]{a/b}$ 31
3.3. Thue's First Result 32
3.4. Effectiveness 33
3.5. Effective Analogues of Theorem 1.6 34
3.6. The First Effective Inequalities of Baker 36
3.7. Effective Bounds on Linear Forms in Algebraic Numbers ... 39
§4. Stronger and More General Versions of Liouville's Theorem and Thue's Theorem 40
4.1. The Dirichlet Pigeonhole Principle 40
4.2. Thue's Method in the General Case 41
4.3. Thue's Theorem on Approximation of Algebraic Numbers .. 44
4.4. The Non-effectiveness of Thue's Theorems 45
§5. Further Development of Thue's Method 45
5.1. Siegel's Theorem 45
5.2. The Theorems of Dyson and Gel'fond 48
5.3. Dyson's Lemma 50
5.4. Bombieri's Theorem 51
§6. Multidimensional Variants of the Thue–Siegel Method 53
6.1. Preliminary Remarks 53
6.2. Siegel's Theorem 53
6.3. The Theorems of Schneider and Mahler 54
§7. Roth's Theorem 55
7.1. Statement of the Theorem 55
7.2. The Index of a Polynomial 56
7.3. Outline of the Proof of Roth's Theorem 57
7.4. Approximation of Algebraic Numbers by Algebraic Numbers 60
7.5. The Number k in Roth's Theorem 61
7.6. Approximation by Numbers of a Special Type 61
7.7. Transcendence of Certain Numbers 62
7.8. The Number of Solutions to the Inequality (62) and Certain Diophantine Equations 63
§8. Linear Forms in Algebraic Numbers and Schmidt's Theorem 65
8.1. Elementary Estimates 65
8.2. Schmidt's Theorem 66
8.3. Minkowski's Theorem on Linear Forms 67
8.4. Schmidt's Subspace Theorem 68

8.5. Some Facts from the Geometry of Numbers 71
§9. Diophantine Equations with the Norm Form 73
9.1. Preliminary Remarks 73
9.2. Schmidt's Theorem 75
§10. Bounds for Approximations of Algebraic Numbers in Non-archimedean Metrics 76
10.1. Mahler's Theorem 76
10.2. The Thue–Mahler Equation 76
10.3. Further Non-effective Results 77

Chapter 2. Effective Constructions in Transcendental Number Theory 78

§1. Preliminary Remarks 78
1.1. Irrationality of e 78
1.2. Liouville's Theorem 79
1.3. Hermite's Method of Proving Linear Independence of a Set of Numbers 80
1.4. Siegel's Generalization of Hermite's Argument 80
1.5. Gel'fond's Method of Proving That Numbers Are Transcendental 82
§2. Hermite's Method 83
2.1. Hermite's Identity 84
2.2. Choice of $f(x)$ and End of the Proof That e is Transcendental 85
2.3. The Lindemann and Lindemann–Weierstrass Theorems 88
2.4. Elimination of the Exponents 90
2.5. End of the Proof of the Lindemann–Weierstrass Theorem 92
2.6. Generalization of Hermite's Identity 92
§3. Functional Approximations 93
3.1. Hermite's Functional Approximation for e^z 93
3.2. Continued Fraction for the Gauss Hypergeometric Function and Padé Approximations 95
3.3. The Hermite–Padé Functional Approximations 98
§4. Applications of Hermite's Simultaneous Functional Approximations 99
4.1. Estimates of the Transcendence Measure of e 99
4.2. Transcendence of e^π 100
4.3. Quantitative Refinement of the Lindemann–Weierstrass Theorem 103
4.4. Bounds for the Transcendence Measure of the Logarithm of an Algebraic Number 104
4.5. Bounds for the Irrationality Measure of π and Other Numbers 106
4.6. Approximations to Algebraic Numbers 110

§5. Bounds for Rational Approximations of the Values of the Gauss Hypergeometric Function and Related Functions ... 112
5.1. Continued Fractions and the Values of e^z ... 112
5.2. Irrationality of π ... 113
5.3. Maier's Results ... 115
5.4. Further Applications of Padé Approximation ... 116
5.5. Refinement of the Integrals. ... 120
5.6. Irrationality of the Values of the Zeta-Function and Bounds on the Irrationality Exponent. ... 121
§6. Generalized Hypergeometric Functions ... 127
6.1. Generalized Hermite Identities ... 128
6.2. Unimprovable Estimates ... 131
6.3. Ivankov's Construction ... 133
§7. Generalized Hypergeometric Series with Finite Radius of Convergence ... 136
7.1. Functional Approximations of the First Kind ... 136
7.2. Functional Approximations of the Second Kind ... 139
§8. Remarks ... 143

Chapter 3. Hilbert's Seventh Problem ... 146

§1. The Euler–Hilbert Problem ... 146
1.1. Remarks by Leibniz and Euler ... 146
1.2. Hilbert's Report ... 146
§2. Solution of Hilbert's Seventh Problem ... 147
2.1. Statement of the Theorems ... 147
2.2. Gel'fond's Solution ... 147
2.3. Schneider's Solution ... 149
2.4. The Real Case ... 150
2.5. Laurent's Method ... 151
§3. Transcendence of Numbers Connected with Weierstrass Functions ... 152
3.1. Preliminary Remarks ... 152
3.2. Schneider's Theorems ... 153
3.3. Outline of Proof of Schneider's Theorems ... 155
§4. General Theorems ... 157
4.1. Schneider's General Theorems ... 157
4.2. Consequences of Theorem 3.17 ... 158
4.3. Lang's Theorem ... 159
4.4. Schneider's Work and Later Results on Abelian Functions .. 159
§5. Bounds for Linear Forms with Two Logarithms ... 161
5.1. First Estimates for the Transcendence Measure of a^b and $\ln\alpha/\ln\beta$ ... 161
5.2. Refinement of the Inequalities (19) and (20) Using Gel'fond's Second Method ... 163

5.3. Bounds for Transcendence Measures 164
5.4. Linear Forms with Two Logarithms 164
5.5. Generalizations to Non-archimedean Metrics 165
5.6. Applications of Bounds on Linear Forms in Two Logarithms 165
§6. Generalization of Hilbert's Seventh Problem to Liouville Numbers 172
6.1. Ricci's Theorem 172
6.2. Later Results 173
§7. Transcendence Measure of Some Other Numbers Connected with the Exponential Function 173
7.1. Logarithms of Algebraic Numbers 173
7.2. Approximation of Roots of Certain Transcendental Equations 175
§8. Transcendence Measure of Numbers Connected with Elliptic Functions 176
8.1. The Case of Algebraic Invariants 176
8.2. The Case of Algebraic Periods 177
8.3. Values of $\wp(z)$ at Non-algebraic Points 177

Chapter 4. Multidimensional Generalization of Hilbert's Seventh Problem 179

§1. Linear Forms in the Logarithms of Algebraic Numbers 179
1.1. Preliminary Remarks 179
1.2. The First Effective Theorems in the General Case 180
1.3. Baker's Method 182
1.4. Estimates for the Constant in (8) 185
1.5. Methods of Proving Bounds for Λ, Λ_0, and Λ_1 187
1.6. A Special Form for the Inequality 188
1.7. Non-archimedean Metrics 188
§2. Applications of Bounds on Linear Forms 189
2.1. Preliminary Remarks 189
2.2. Effectivization of Thue's Theorem 189
2.3. Effective Strengthening of Liouville's Theorem 192
2.4. The Thue–Mahler Equation 193
2.5. Solutions in Special Sets 194
2.6. Catalan's Equation 195
2.7. Some Results Connected with Fermat's Last Theorem 196
2.8. Some Other Diophantine Equations 197
2.9. The *abc*-Conjecture 199
2.10. The Class Number of Imaginary Quadratic Fields 199
2.11. Applications in Algebraic Number Theory 200
2.12. Recursive Sequences 201
2.13. Prime Divisors of Successive Natural Numbers 203

2.14. Dirichlet Series 203
§3. Elliptic Functions 204
3.1. The Theorems of Baker and Coates 204
3.2. Masser's Theorems 204
3.3. Further Results 205
3.4. Wüstholz's Theorems 206
§4. Generalizations of the Theorems in §1 to Liouville Numbers 207
4.1. Walliser's Theorems 207
4.2. Wüstholz's Theorems 207

Chapter 5. Values of Analytic Functions That Satisfy Linear Differential Equations 209

§1. E-Functions 209
1.1. Siegel's Results 209
1.2. Definition of E-Functions and Hypergeometric E-Functions . 210
1.3. Siegel's General Theorem 213
1.4. Shidlovskii's Fundamental Theorem 214
§2. The Siegel–Shidlovskii Method 215
2.1. A Technique for Proving Linear and Algebraic Independence 215
2.2. Construction of a Complete Set of Linear Forms 218
2.3. Nonvanishing of the Functional Determinant 219
2.4. Concluding Remarks 221
§3. Algebraic Independence of the Values of Hypergeometric E-Functions 222
3.1. The Values of E-Functions That Satisfy First, Second, and Third Order Differential Equations 222
3.2. The Values of Solutions of Differential Equations of Arbitrary Order 225
§4. The Values of Algebraically Dependent E-Functions 229
4.1. Theorem on Equality of Transcendence Degree 230
4.2. Exceptional Points 231
§5. Bounds for Linear Forms and Polynomials in the Values of E-Functions 234
5.1. Bounds for Linear Forms in the Values of E-Functions 234
5.2. Bounds for the Algebraic Independence Measure 237
§6. Bounds for Linear Forms that Depend on Each Coefficient 241
6.1. A Modification of Siegel's Scheme 241
6.2. Baker's Theorem and Other Concrete Results 243
6.3. Results of a General Nature 244
§7. G-Functions and Their Values 246
7.1. G-Functions 246
7.2. Canceling Factorials 250
7.3. Arithmetic Type 252

7.4. Global Relations 253
7.5. Chudnovsky's Results 255

Chapter 6. Algebraic Independence of the Values of Analytic Functions That Have an Addition Law 259

§1. Gel'fond's Method and Results 260
1.1. Gel'fond's Theorems 261
1.2. Bound for the Transcendence Measure 262
1.3. Gel'fond's "Algebraic Independence Criterion" and the Plan of Proof of Theorem 6.3 264
1.4. Further Development of Gel'fond's Method 267
1.5. Fields of Finite Transcendence Type 268
§2. Successive Elimination of Variables 271
2.1. Small Bounds on the Transcendence Degree 271
2.2. An Inductive Procedure 272
§3. Applications of General Elimination Theory 276
3.1. Definitions and Basic Facts 276
3.2. Philippon's Criterion 278
3.3. Direct Estimates for Ideals 284
3.4. Effective Hilbert Nullstellensatz 287
§4. Algebraic Independence of the Values of Elliptic Functions 290
4.1. Small Bounds for the Transcendence Degree 291
4.2. Elliptic Analogues of the Lindemann–Weierstrass Theorem . 295
4.3. Elliptic Generalizations of Hilbert's Seventh Problem 296
§5. Quantitative Results 302
5.1. Bounds on the Algebraic Independence Measure 302
5.2. Bounds on Ideals, and the Algebraic Independence Measure . 304
5.3. The Approximation Measure 306

Bibliography 309

Index 344

Notation

$\mathbb{N}$ is the set of natural numbers

$\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$

$\mathbb{Z}$ is the set of integers

$\mathbb{Q}$ is the set of rational numbers

$\mathbb{R}$ is the set of real numbers

$\mathbb{C}$ is the set of complex numbers

$\mathbb{A}$ is the set of algebraic numbers

$\mathbb{Z}_{\mathbb{A}}$ is the set of all algebraic integers

$\mathbb{Z}_{\mathbb{Z}}$ is the set of all algebraic integers of the field $\mathbb{K}$

$\mathbb{K}(z_1, \ldots, z_m)$ is the set of all rational functions in the variables $z_1, \ldots, z_m$ over the field $\mathbb{K}$

$\mathbb{K}[z_1, \ldots, z_m]$ is the set of all polynomials in the variables $z_1, \ldots, z_m$ over the field $\mathbb{K}$

$H(P(z)) = H(P)$ is the height of the polynomial $P(z) \in \mathbb{C}[z_1, \ldots, z_m]$, i.e., the maximum absolute value of its coefficients

$L(P(z)) = L(P)$ is the length of the polynomial $P(z) \in \mathbb{C}[z_1, \ldots, z_m]$, i.e., the sum of the absolute values of its coefficients

$\deg_{z_i} P$ is the degree in z_i of the polynomial P

$\deg P$ is the total degree of the polynomial P

$t(P) = \deg P + \ln H(P)$

$h(I)$ is the rank of the homogeneous ideal $I \subset \mathbb{Z}[x_0, \ldots, x_m]$

$\deg I$ is the degree of the homogeneous ideal $I \subset \mathbb{Z}[x_0, \ldots, x_m]$

$H(I)$ is the height of the homogeneous ideal $I \subset \mathbb{Z}[x_0, \ldots, x_m]$

$t(I) = \deg I + \ln H(I)$

$|I(\omega)|$ is the magnitude of the homogeneous ideal $I \subset \mathbb{Z}[x_0, \ldots, x_m]$ at the point $\omega \in \mathbb{C}^{m+1}$

$\deg \alpha$ is the degree of the algebraic number α

$H(\alpha)$ is the height of the algebraic number α

$L(\alpha)$ is the length of the algebraic number α

$\mathrm{Norm}(\alpha)$ is the product of all of the conjugates of the algebraic number α

$\lceil \alpha \rceil$ is the maximum absolute value of the conjugates of the algebraic number α

$|\alpha|_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic norm of the algebraic number α

$\|a\|$ is the distance from the number $a \in \mathbb{R}$ to the nearest integer

$\|x\| = \max_{1 \le i \le m} |x_i|$ is the sup-norm of the vector $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$

$[a]$ is the greatest integer function of the real number a

$\mathrm{tr}\deg \mathbb{K}$ is the transcendence degree of the field $\mathbb{K}$

$\tau(\mathbb{K})$ is the transcendence type of the field $\mathbb{K} \subset \mathbb{C}$

$s(a)$ is the size of the complex number a

$\mu(a)$ is the exponent of irrationality of the real number a