

Calculus

Fong Yuen • Wang Yuan



Springer

方積微

杜忠誥書



方源 王元

Fong Yuen
Department of Mathematics
National Cheng Kung University
Tainan
Taiwan 701

Wang Yuan
Institute of Mathematics
Academia Sinica
Beijing 100080
China

Library of Congress Cataloging-in-Publication Data

Fong, Yuen.

Calculus / Yuen Fong, Yuan Wang.

p. cm.

Includes index.

ISBN 9813083018 (softcover) (Singapore)

9813083522 (hardcover)

1. Calculus I. Wang, Yuan, 1930- . II. Title

QA 303.F586 1999

515--dc20

96-29408

CIP

ISBN 981-3083-01-8 (softcover)

ISBN 981-3083-52-2 (hardcover)

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on micro-films or in any other way, and storage in databanks or in any system now known or to be invented. Permission for use must always be obtained from the publisher in writing.

© Springer-Verlag Singapore Pte. Ltd. 2000

The publisher makes no representation, express or implied, with regard to the accuracy of the information contained in this book and cannot accept any legal responsibility or liability for any errors or omissions that may be made.

Typesetting: Asco Typesetters, Hong Kong
Printed in Singapore

SPIN 10546278 (softcover) 5 4 3 2 1 0

SPIN 10748684 (hardcover) 5 4 3 2 1 0

Springer

Singapore

Berlin

Heidelberg

New York

Barcelona

Hong Kong

London

Milan

Paris

Tokyo

Calculus

Fong Yuen • Wang Yuan



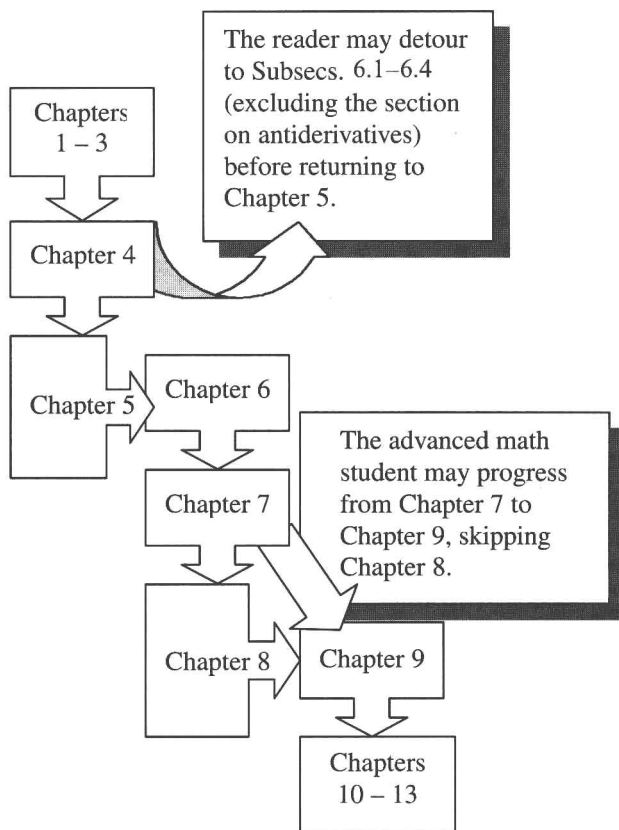
Springer

Preface

The text presented here is designed for a two-semester course on elementary calculus. It has been used in the past 20 years by students in various scientific disciplines throughout Taiwan and mainland China. The main aim of writing this book is to make it accessible to all students—those familiar with the subject or not—but the arguments herein are rigorous. However, attempts have been made to present the main philosophy and thinking behind calculus through various innovative examples and applications so that the abstract notions and concepts of calculus are within reasonable understanding.

Reading and understanding mathematics is an art that is completely different from reading any other literature. Readers need practice reading mathematics. Therefore, a solutions manual containing detailed solutions to all exercises in this book will be provided separately. This hopefully will assist students engaged in self-study or who are not under formal instruction.

Recommended Usage of "Calculus"



Acknowledgements

At this point, the first author would like to credit those having an indelible and positive influence on his mathematical development and career. Among his best and most influential teachers and friends include Professors I.H. Lin, National Cheng Kung University; J.D.P. Meldrum, University of Edinburgh; A. Oswald, University of Teesside; G. Pilz, Johannes Kepler Universität Linz; the late James R. Clay, University of Arizona; Li Fu-an, Chinese Academy of Sciences; R. Wiegandt, Hungarian Academy of Sciences; K. Kaarli, University of Tartu; K.I. Beidar, Wen-Fong Ke, C.K. Lin, Lam Ngau, and Cheng-I Weng, National Cheng Kung University; A.V. Mikhalev, Moscow State University; Yeong-Nan Yeh, Academia Sinica; and Y.H. Xu, Fudan University. Special mention should go to James R. Clay ("Big Jim"), who the first author the laws of thought and philosophy which are of utmost importance for a learned mathematician. Section 5.6 is dedicated to him for their lifelong academic collaboration.

Further, this book would not come into fruition without the help of many others. Here, acknowledgement should go to those who have encouraged and supported the first author financially and spiritually through the years. These include Fong Ka Wai, Emily Fong Cheng Sau Kwan, Ching-Kwong Chow, Dunnet S.W. Cheung, Lau Fung, William Tau W.Y., Abel Tsui Chun Kong, Ng Kwok Ying, Raleigh Siu Lau, Samuel Lee Wai Hung, Wong Yiu Cheong, Chak Chung Man, and Tina Wong. However, special mention must be made to the first author's brother, Fong Ka wai, who has been supportive of the former's academic career for the past 20 years. Both him and Ching-Kwong Chow have generously purchased and donated copies of this book to university libraries in mainland China. This is in response to the event, the **Hope of Education** for the 21st century, in China.

On the preparation of this book, sincere thanks go to Lee Jen Yea who devoted countless hours typing the manuscript. Thanks are also due to

Professors Long-Shung Shiao, Shian Leou, Y.S.Tsai, H.J. Shyr, J.S. Lin, P.H. Lee, Min-te Chao, and Liao Lung-Chi who have been supportive of this book. Also, the authors are most grateful to Professors Chao and Miao for the use of their figures, especially in the last two chapters.

Finally, gratitude must be given to Maurice Kwong, Managing Director, and Ian Shelley, former Director of Publishing, Springer-Verlag Singapore, for their assistance in the publication of this book.

Fong Yuen	Wang Yuan
Tainan	Beijing

March 1999

Contents

	Preface	v
	Acknowledgements	vii
Chapter 1	Introduction	1
	1.1 What is Calculus?	1
	1.2 Sets and Functions	5
	1.3 System of Numbers	18
	1.4 Mathematical Induction	27
	1.5 Analytic Plane Geometry	30
Chapter 2	Limit and Continuity	41
	2.1 The Concept of Limit	41
	2.2 Some Limit Theorems	64
	2.3 Continuity	82
	2.4 Some Theorems on Continuous Functions	93
	2.5 Uniform Continuity	97
Chapter 3	Differentiation	99
	3.1 Some Definitions of Derivatives	99
	3.2 Some Formulas of Derivatives	111
	3.3 The Chain Rule	120
	3.4 Derivatives of Trigonometric Functions	128
	3.5 Implicit Differentiation and Higher Derivatives	137
	3.6 Differentials and Newton-Raphson Approximations	149

Chapter 4	Application of Derivatives	159
4.1	Rolle's Theorem and Mean Value Theorem	159
4.2	Monotonic Functions	166
4.3	Relative Extrema of a Function	168
4.4	Convexity of a Function	176
4.5	Some Graph Sketching	195
Chapter 5	Integration	203
5.1	An Area Problem	203
5.2	Definition of Definite Integral	208
5.3	Some Theorems on Integral Calculus	229
5.4	Fundamental Theorem of Calculus	237
5.5	Area Between Graphs	248
5.6	Application: Generalizations of Pythagoras' Theorem	255
5.7	More on Applications	266
Chapter 6	Some Special Functions	293
6.1	Inverse Functions	293
6.2	Inverse Trigonometric Functions	300
6.3	Exponential and Logarithm Functions	315
6.4	Hyperbolic and Inverse Hyperbolic Functions	335
Chapter 7	Formal Integration	349
7.1	Indefinite Integrals	349
7.2	Integration by Parts	356
7.3	Trigonometric Integrals	368
7.4	Integration by Trigonometric Substitution and Integrands Involving $\sqrt{x^2 - a^2}$ and $\sqrt{a^2 \pm x^2}$	378
7.5	Partial Fractions	390
Chapter 8	Numerical Integration	399
8.1	Trapezoidal Rule	399
8.2	Simpson's Rule	408
Chapter 9	More on Limits and Improper Integrals	417
9.1	Sequences of Real Numbers and Limits of Sequences	417
9.2	Some Important Limits	429
9.3	L'Hôpital's Rules for Indeterminate Forms	433
9.4	Improper Integrals	446

Chapter 10 Infinite Series	461
10.1 Infinite Series	461
10.2 Positive-term Series: Comparison Test and Integral Test	473
10.3 Alternating Series, Absolute Convergence, and Ratio and Root Tests	481
10.4 Power Series, Maclaurin Series, and Taylor Series	495
Chapter 11 Polar Coordinates	513
11.1 Polar Coordinates	513
11.2 Areas in Polar Coordinates	525
11.3 Parametric Paths and Lengths	530
Chapter 12 Differential Calculus for Functions of Several Variables	535
12.1 Function of n -variables	535
12.2 Partial Derivatives	545
12.3 Limits and Continuity	557
12.4 The Chain Rule	571
12.5 Gradients and Directional Derivatives	589
12.6 Implicit Differentiation	614
12.7 Extrema for Functions of Several Variables	619
Chapter 13 Multiple Integrals	637
13.1 Double Integrals Over a Rectangle	637
13.2 Double Integrals Over a Region	653
13.3 Double Integrals in Polar Coordinates	669
13.4 Triple Integrals and Their Applications	680
Tables	699
Answers to Selected Exercises	703
Index	793

1

Introduction

1.1 What is Calculus?

To today's scientists, calculus is a kind of elementary mathematics (say, algebra, geometry, and trigonometry) enhanced by the **limiting process**, while to people of ancient Rome, calculus was a pebble that was used in gambling and counting. Actually, calculus is the study of the behavior of real functions of real variables by means of a basic notion called **limit**. This very idea of limit immediately leads to the concepts of differentiation and integration, which provide numerous applications in the various disciplines. For example, a physicist may use integral calculus to determine the work done by a variable force, a chemist may employ differential calculus to investigate the results of various chemical reactions, a biologist may use calculus to forecast the outcome of the rate of growth of bacteria in a culture, and finally, an economist may apply it to problems that involve corporate profits and losses.

Calculus obtains its ideas from elementary mathematics and extends them to more general situations. Table 1.1.1 gives a contrast between pre-calculus (elementary mathematics without calculus) and calculus.

Now we shall give a brief account of the history of calculus. Formally speaking, the actual invention of calculus is due to two eminent mathematicians, Sir Isaac Newton (1642–1727) of England and Gottfried Wilhelm von Leibniz (1646–1716) of Germany. However, one can trace back to the time of ancient Greece where the original notion of calculus was found by Archimedes in the area problem about 2500 years ago. By using the “method of exhaustion”, the Greeks were able to determine the area of any polygon by dividing it into triangles as shown in Figure 1.1.1 and adding the areas of these triangles to form the required area A .

It is a more difficult task to determine the area of a curved region. The Greek's method of exhaustion was to inscribe polygons in a curved region and then let the number of sides of polygons increase.

Table 1.1.1

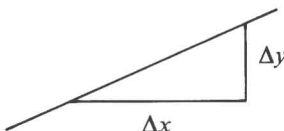
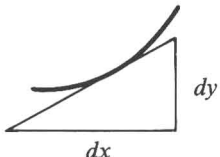
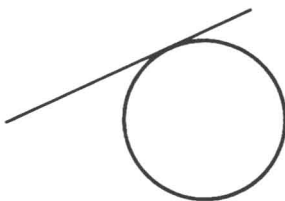

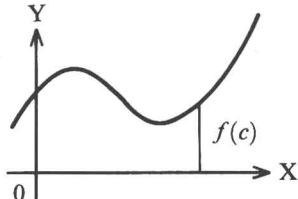
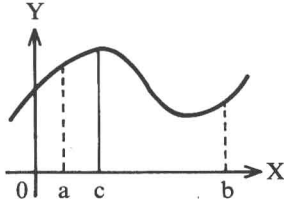
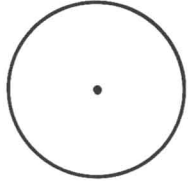
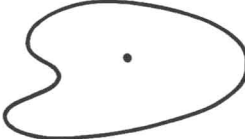
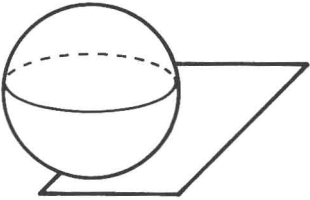
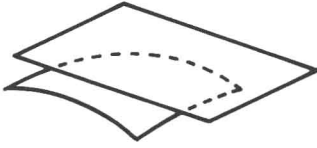


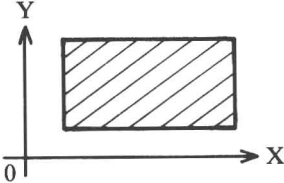
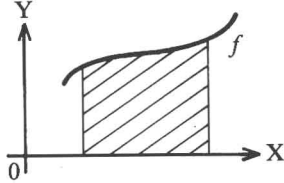
Precalculus	Calculus
 <p>m: slope of a straight line $y = mx + c$ where $m = \Delta y / \Delta x$</p>	 <p>$f'(x)$: slope of a curve $y = f(x)$ where $f'(x) = dy/dx$</p>
 <p>tangent line to a circle</p>	 <p>tangent line to a general curve</p>
 <p>$f(c)$: height of a curve at c</p>	 <p>$f(c)$: maximum height of a curve at $c \in [a, b]$</p>
 <p>center of a circle</p>	 <p>centroid of a region</p>

Table 1.1.1
(cont.)

Precalculus	Calculus
 <p>tangent plane to a sphere</p>	 <p>tangent plane to a general surface</p>
$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$ <p>sum of finite numbers</p>	$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n + \cdots$ <p>sum of infinite numbers</p>
 <p>length of a line segment</p>	 <p>length of an arc</p>
 <p>area of a rectangle</p>	 <p>area under a curve</p>

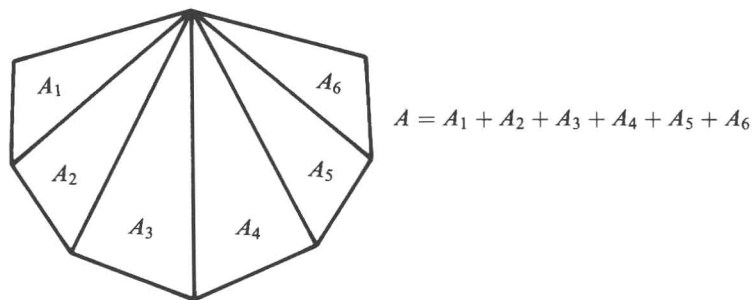


Figure 1.1.1

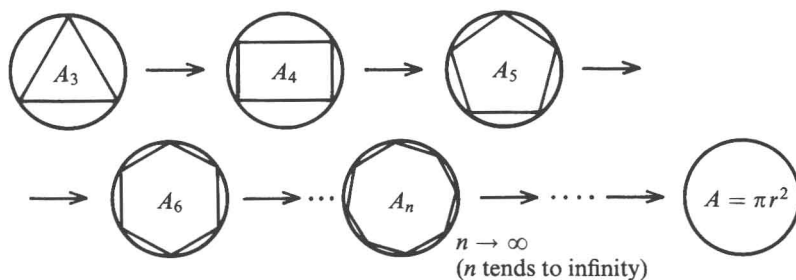


Figure 1.1.2

Figure 1.1.2 shows that the actual area of a circle is in fact derivable by taking a limiting process of inscribed regular polygons, i.e., let A_n be the area of the inscribed regular polygon with n sides, and as n increases, the area of A_n becomes closer to the area of a circle. Here we say that the area of the circle is equal to the **limit** of the areas of the inscribed regular polygons and denote it as

$$A = \lim_{n \rightarrow \infty} A_n.$$

One should notice that not only the ancient Greeks used the concept of limit in an ambiguous manner, even Newton and Leibnitz developed their ideas of the derivative and integral with only a vague notion of the limit concept. It was not until more than a century later that Augustin-Louis Cauchy (1789–1857), a famous French mathematician, then gave a precise definition for the limit of a function that we all use today. Thus, the results of calculus can be proved in a rigorous manner and have been universally accepted since the 19th century.

1.2 Sets and Functions

Although an excursion into the theory of sets is not necessarily needed at this stage, an intuitive understanding of the basics of set theory will not only be helpful throughout the study of the present text but will also facilitate mathematical communications with your fellow classmates and professors. Here we would like to emphasize that the word *set* is the accepted technical term for such synonyms as *class*, *collection*, *aggregate*, and *assemblage*. In daily life, we use such phrases as a set of pictures, a set of books, a set of programmes, and so on. However, in this book, we are more interested in sets of collections of mathematical objects, such as a *set of real numbers*, a *set of positive integers*, a *set of points in the X - Y plane*, and so on. Thus, all these examples give us an intuitive feeling that a set is a collection of well-defined objects. So, we have the following definition.

Definition 1.2.1 Definition of a Set

A set is a collection of well-defined objects (points, elements, or members).

We note that the above definition permits us to form the set of all the elements x which satisfy certain constrained conditions or properties, namely $P(x)$, except for the *set of all the sets x which satisfy $P(x)$* . This distinction suffices to eliminate the logical paradoxes. Once the semantic paradoxes have been avoided, the set S whose existence is asserted by Definition 1.2.1 will be designated by the symbol

$$S = \{x|P(x)\}.$$

The above expression is read as “ S is the set that contains the element x such that x satisfies the property $P(x)$ ”.

We generally designate sets by capital letters X, Y, Z , etc., and elements of a set by small letters x, y, z , etc. If a set S contains the points x, y, z, w , we write

$$S = \{x, y, z, w\}.$$

The notation “ $x \in S$ ” means “ x is an element of S ” or “ x belongs to S ”; the notation “ $x \notin S$ ” means “ x is not an element of S ” or “ x does not belong to S ”.

Example 1.2.2 Examples of Sets

- (i) $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ is the *set of all natural numbers* (or *positive integers*).