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VOLUME 23

Geometrical Theory of Dynamical Systems and Fluid Flows

Revised Edition

Tsutomu Kambe

World Scientific

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Preface to Revised Edition

Since the first edition in 2004, it has been five years. This book is an introductory text on geometrical theory of dynamical systems, fluid flows, and certain integrable systems. The aim is to unify two different subjects of *Geometry* and *Dynamics*. The former is a mathematical subject regarded traditionally as being applied to static objects, while the latter is a physical subject describing time evolution of mechanical systems.

In the author's original idea, Part III of this book was meant to be the central part that describes *Fluid Mechancis*, a field to which the author had devoted himself for many years. But during the preparation of the book, he became aware of other areas that are also equally important to learn the *interplay* between geometry and dynamics. Part I on *Mathematical Bases* is helpful to understand the background in depth.

After the first publication, the topic in Chapter 7 has been improved significantly. So that, in this revised edition, Chapter 7 is rewritten completely, to describe variational formulation of ideal fluid flows in the light of modern gauge theory of theoretical physics. In particular, §7.12 describes a new formulation of *fluid* Maxwell equations. Using this opportunity, the beginning sections of Chapter 8 have been rewritten in order for the readers to easily access this unfamiliar approach to fluid flows. Also, §3.7 of Chapter 3 is improved to clarify the original idea of Arnold (1966). New Appendices I and J are added (by replacing the old I with the new I). The author has tried his best to improve and correct descriptions of all the other chapters too, and wishes that the book will be well received by all readers.

T. Kambe
May 2009

Preface to First Edition

This is an introductory textbook on the geometrical theory of dynamical systems, fluid flows, and certain integrable systems. The subjects are interdisciplinary and extend from mathematics, mechanics and physics to mechanical engineering. The approach is very fundamental and would be traced back to the times of Poincaré, Weyl and Birkhoff in the first half of the 20th century. The theory gives geometrical and frame-independent characterizations of various dynamical systems and can be applied to chaotic systems as well from the geometrical point of view. For integrable systems, similar but different geometrical theory is presented.

Underlying concepts of the present subject are based on the *differential geometry* and the *theory of Lie groups* in mathematical aspect and based on the *gauge theory* in physical aspect. Usually, those subjects are not easy to access, nor familiar to most students in physics and engineering. A great deal of effort has been directed to make the description elementary, clear and concise, so that beginners have easy access to the subject. This textbook is intended for upper level undergraduates and postgraduates in physics and engineering sciences, and also for research scientists interested in the subject.

Various dynamical systems often have common geometrical structures that can be formulated on the basis of Riemannian geometry and Lie group theory. Such a dynamical system always has a symmetry, namely it is invariant under a group of transformations, and furthermore it is necessary that the group manifold is endowed with a Riemannian metric. In this book, pertinent mathematical concepts are illustrated and applied to physical problems of several dynamical systems and integrable systems.

The present text consists of four parts: I. *Mathematical Bases*, II. *Dynamical Systems*, III. *Flows of Ideal Fluids*, and IV. *Geometry of Integrable Systems*. Part I is composed of three chapters where basic mathematical concepts and tools are described. In Part II, three dynamical systems are presented in order to illustrate the fundamental idea on the basis of the mathematical framework of Part I. Although those systems are well-known in mechanics and physics, new approach and formulation will be provided from a geometrical point of view. Part III includes two new theoretical formulations of flows of ideal fluids: one is a variational formulation on the basis of the gauge principle and the other is a geometrical formulation based on a group of diffeomorphisms and associated Riemannian geometry. Part IV aims at presenting a different geometrical formulation for integrable systems. Its historical origin is as old as the Riemannian geometry and traced back to the times of Bäcklund, Bianchi and Lie, although modern theory of geometry of integrable systems is still being developed.

More details of each Part are as follows. In Part I, before considering particular dynamical systems, mathematical concepts are presented and reviewed concisely. In the first chapter, basic mathematical notions are illustrated about flows, diffeomorphisms and the theory of Lie groups. In the second chapter, the geometry of surface in Euclidian space $\mathbb{R}^3$ is summarized with special emphasis on the Gaussian cuvature which is one of the central objects in this treatise. This chapter presents many elementary concepts which are developed subsequently. In the third chapter, theory of Riemannian differential geometry is summarized concisely and basic concepts are presented: the first and second fundamental forms, commutator, affine connection, geodesic equation, Jacobi field, and Riemannian curvature tensors.

The three dynamical systems of Part II are fairly simple but fundamental systems known in mechanics. They were chosen to illustrate how the geometrical theory can be applied to dynamical systems. The first system in Chapter 4 is a free rotation of a rigid body (Euler's top). This is a well-known problem in physics and one of the simplest nonlinear integrable systems of finite degrees of freedom. Chapter 5 illustrates derivation of the KdV equation as a geodesic equation on a group (actually an extended group) of diffeomorphisms, which gives us a geometrical characterization of the KdV system. The third example in Chapter 6 is a geometrical analysis of chaos of a Hamiltonian system, which is a self-gravitating system of a finite number of point masses.

Part III is devoted to Fluid Mechanics which is considered to be a central part of the present book. In Chapter 7, a new gauge-theoretical formulation is presented, together with a consistent variational formulation in terms of variation of material particles. As a result, Euler's equation of motion is derived for an isentropic compressible flow. This formulation implies that the vorticity is a gauge field. Chapter 8 is a Riemannian-geometrical formulation of the hydrodynamics of an incompressible ideal fluid. This gives us not only geometrical characterization of fluid flows but also interpretation of the origin of Riemannian curvatures of flows. Chapter 9 is a geometrical formulation of motions of a vortex filament.

It is well known that some soliton equations admit a geometric interpretation. In Part IV, Chapter 10 reviews a classical theory of the sine-Gordon equation and the Bäcklund transformation which is an oldest example of geometry of a pseudo-spherical surface in $\mathbb{R}^3$ with the Gaussian curvature of a constant negative value. Chapter 11 presents a geometric and group-theoretic theory for integrable systems such as sine-Gordon equation, nonlinear Schrodinger equation, nonlinear sigma model and so on. Final section presents a new finding [CFG00] that all integrable systems described by the $su(2)$ algebra are mapped to a spherical surface.

Highlights of this treatise would be: (i) Geometrical formulation of dynamical systems; (ii) Geometric description of ideal-fluid flows and an interpretation of the origin of Riemannian curvatures of fluid flows; (iii) Various geometrical characterizations of dynamical fields; (iv) Gauge-theoretic description of ideal fluid flows; and (v) Modern geometric and group-theoretic formulation of integrable systems.

It is remarkable that the present geometrical formulations are successful for all the problems considered here and give insight into common background of the diverse physical systems. Furthermore, the geometrical formulation opens a new approach to various dynamical systems.

Parts I–III of the present monograph were originally prepared as *lecture notes* during the author's stay at the Isaac Newton Institute in the programme "Geometry and Topology of Fluid Flow" (2000). After that, the manuscript had been revised extensively and published as a *Review* article in the journal, *Fluid Dynamics Research*. In addition, the present book includes Part IV, which describes geometrical theory of *Integrable Systems*. Thus, this covers an extensive area of dynamical systems and reformulates those systems on the basis of *geometrical concepts*.

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December 2003

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