ERGEBNISSE DER MATHEMATIK UND IHRER GRENZGEBIETE

UNTER MITWIRKUNG DER SCHRIFTLEITUNG DES „ZENTRALBLATT FÜR MATHEMATIK“

NEUE FOLGE · HEFT 28

CLUSTER SETS

BY

KIYOSHI NOSHIRO

SPRINGER-VERLAG
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REIHE:

MODERNE FUNKTIONENTHEORIE

BESORGT VON

L. V. AHLFORS

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Preface

For the first systematic investigations of the theory of cluster sets of analytic functions, we are indebted to IVERSEN [1—3] and GROSS [1—3] about forty years ago. Subsequent important contributions before 1940 were made by SEIDEL [1—2], DOOB [1—4], CARTWRIGHT [1—3] and BEURLING [1]. The investigations of SEIDEL and BEURLING gave great impetus and interest to Japanese mathematicians; beginning about 1940 some contributions were made to the theory by KUNUGUI [1—3], IRIE [1], TÔKI [1], TUMURA [1—2], KAMETANI [1—4], TSUJI [4] and NOSHIRO [1—4]. Recently, many noteworthy advances have been made by BAGEMIHL, SEIDEL, COLLINGWOOD, CARTWRIGHT, HERVÉ, LEHTO, LOHWATER, MEIER, OHTSUKA and many other mathematicians. The main purpose of this small book is to give a systematic account on the theory of cluster sets.

Chapter I is devoted to some definitions and preliminary discussions. In Chapter II, we treat extensions of classical results on cluster sets to the case of single-valued analytic functions in a general plane domain whose boundary contains a compact set of essential singularities of capacity zero; it is well-known that HÄLLSTRÖM [2] and TSUJI [7] extended independently Nevanlinna's theory of meromorphic functions to the case of a compact set of essential singularities of logarithmic capacity zero. Here, Ahlfors' theory of covering surfaces plays a fundamental rôle. Chapter III is concerned with functions meromorphic in the unit circle. We discuss here functions of class (U) in Seidel's sense, boundary theorems of COLLINGWOOD-CARTWRIGHT, recent important results of BAGEMIHL-SEIDEL and COLLINGWOOD on the relation between Baire category and cluster sets, Bagemihl's results on ambiguous points, Meier's results related to Lusin-Privaloff-Plessner's theorem and results of LEHTO and VIRTANEN on meromorphic functions of bounded type and normal meromorphic functions. In Chapter IV, we deal with single-valued analytic functions on open Riemann surfaces and discuss covering properties and boundary behaviours. We state here some recent results of HEINS, KURODA, KURAMOCHI and CONSTANTINESCU-CORNEA from the view-point of cluster sets. We hope that these fragmentary treatments will contribute to the future theory of cluster sets of analytic functions on open Riemann surfaces. Appendix is devoted to cluster sets of pseudo-analytic functions. A recent paper of BEURLING-AHLFORS [1] contains a striking result from the view-point of cluster sets. We cannot apply the theory of functions of class (U) in Seidel's sense to the case of pseudo-analytic functions without any additional condition. We discuss to what extent results on cluster sets of analytic functions can be extended to the case of pseudo-analytic functions.

It has been my earnest desire to write a systematic account on cluster sets since some years ago. I should like to express my hearty thanks to Professor LARS V. AHLFORS for his kind recommendation to the Ergebnisse Series. I am very grateful to my colleague Professor T. KURODA and my students Mr. R. IWAHASHI, Mr. M. KISHI and Mr. M. NAKAI for their careful readings of the manuscript and for their helpful comments. It is also a pleasure to acknowledge the constant generosity and courtesy of the Springer Verlag.

November 10, 1959 KIYOSHI NOSHIRO
Nagoya

Contents

I. Definitions and preliminary discussions

§ 1. Definitions of Cluster Sets

1. Let D be an arbitrary domain[1] with boundary Γ. Let E be a totally disconnected closed set contained in Γ. We suppose that $w = f(z)$ is non-constant, single-valued and meromorphic in D. We associate with every point z_0 of Γ the following sets of values.

(i) The **Cluster Set** $C_D(f, z_0)$. $\alpha \in C_D(f, z_0)$ if there exists a sequence of points $\{z_n\}$ with the following properties

$$z_n \in D\,, \quad \lim_{n\to\infty} z_n = z_0\,, \quad \lim_{n\to\infty} f(z_n) = \alpha\,. \tag{1}$$

If we denote by $\mathfrak{D}_r$ the set of values of $w = f(z)$ in the intersection D_r of D with a circular disc $|z - z_0| < r$, then

$$C_D(f, z_0) = \bigcap_{r>0} \overline{\mathfrak{D}}_r\,, \tag{2}$$

where $\overline{\mathfrak{D}}_r$ denotes the closure of $\mathfrak{D}_r$[2].

Evidently $C_D(f, z_0)$ is a non-empty closed set. In the particular case where D is a Jordan domain bounded by a simple closed curve, $C_D(f, z_0)$ is either a single point or a continuum. However, this property does not hold in general cases[3].

Remark. Consider the special case where z_0 is an accessible boundary point of D. Then, there exists a path (simple curve) L in D terminating at z_0. Denote by z_r the last point of intersection of L with a circle c: $|z - z_0| = r$ and by L_r the arc $\widehat{z_r, z_0}$ of L; such an arc is called a last part of L. The intersection D_r of D with (c): $|z - z_0| < r$ is an open set which consists of at most an enumerably infinite number of connected components. Let Δ_r be the component which contains the last part L_r of L. If we denote by $\mathfrak{D}_r^*$ the value set of $f(z)$ in Δ_r, then $\mathfrak{D}_r^*$ is a domain and, hence, $\overline{\mathfrak{D}_r^*}$ is a continuum. Hence the set

$$C_D(f, z_0; L) = \bigcap_{r>0} \overline{\mathfrak{D}_r^*} \tag{3}$$

is either a single point or a continuum. Suppose that L' is another path in D terminating at z_0. If, for every sufficiently small $r(>0)$, the last

[1] The order of connectivity of D may be infinite.

[2] For any point set M, $\overline{M}$ always denotes its closure.

[3] Take as D the unit circular disc with a radial slit and select a boundary point $z_0(\neq 0)$ on the slit. If $w = f(z)$ is a function mapping D conformally onto $|w| < 1$, then $C_D(f, z_0)$ consists of two points.

parts L_r and L'_r can be joined by a suitable path in D_r, then we say that L and L' are equivalent and define the same accessible boundary point of D at z_0. It is easy to show that if L and L' are equivalent, then

$$C_D(f, z_0; L) = C_D(f, z_0; L') \,. \tag{4}$$

Evidently

$$C_D(f, z_0; L) \subset C_D(f, z_0) \,. \tag{5}$$

In the particular case where D is a Jordan domain,

$$C_D(f, z_0) = C_D(f, z_0; L) \,. \tag{6}$$

(ii) The **Boundary Cluster Sets** $C_\Gamma(f, z_0)$ and $C_{\Gamma - E}(f, z_0)$. $\alpha \in C_\Gamma(f, z_0)$ [resp. $C_{\Gamma - E}(f, z_0)$] if there exists a sequence of points $\{\zeta_n\}$ of $\Gamma - z_0$ [resp. $\Gamma - z_0 - E$] such that

$$w_n \in C_D(f, \zeta_n) \text{ for each } n,$$
$$z_0 = \lim_{n\to\infty} \zeta_n \quad \text{and} \quad \alpha = \lim_{n\to\infty} w_n;$$

i. e., if M_r denotes the closure of the union $\bigcup_\zeta C_D(f, \zeta)$ for every ζ of the common part of $\Gamma - z_0$ [resp. $\Gamma - z_0 - E$] and (c): $|z - z_0| < r$, then $\bigcap_{r>0} M_r$ is $C_\Gamma(f, z_0)$ [resp. $C_{\Gamma - E}(f, z_0)$]. Obviously $C_\Gamma(f, z_0)$ and $C_{\Gamma - E}(f, z_0)$ are closed;

$$C_{\Gamma - E}(f, z_0) \subset C_\Gamma(f, z_0) \subset C_D(f, z_0) \,; \tag{7}$$

if $z_0 \in \Gamma - E$ or if $z_0 \in E - E'$, E' denoting the derived set of E, then

$$C_{\Gamma - E}(f, z_0) = C\;(f, z_0) \,.$$

$C_\Gamma(f, z_0)$ is empty if and only if z_0 is an isolated boundary point; $C_{\Gamma - E}(f, z_0)$ is empty if and only if $z_0 \notin \overline{(\Gamma - E)}$.

(iii) The **Range of Values** $R_D(f, z_0)$. This is defined as the set of values α such that $z_n \in D$, $\lim_{n\to\infty} z_n = z_0$, $f(z_n) = \alpha$; i. e.,

$$R_D(f, z_0) = \bigcap_{r>0} \mathfrak{D}_r \,, \tag{8}$$

where $\mathfrak{D}_r$ is the value set of $w = f(z)$ in the common part of D and (c): $|z - z_0| < r$. Accordingly, $R_D(f, z_0)$ is a G_δ set.

(iv) The **Asymptotic Set** $A_D(f, z_0)$. Let z_0 be an accessible boundary point of D. A complex number α is called an *asymptotic value* of $w = f(z)$ at z_0 if $f(z) \to \alpha$ as $z \to z_0$ along a path in D terminating at z_0. The asymptotic set $A_D(f, z_0)$ is defined as the set of asymptotic values of $f(z)$ at z_0. We define $A_D(f, z_0) = \theta$ when z_0 is an inaccessible boundary point, for the sake of convenience.

2. We shall state a relation between $C_D(f, z_0; L)$ and $C_{\Gamma - E}(f, z_0)$ which will be used later. If z_0 is an accessible boundary point of D defined

by a path L in D terminating at z_0 and if z_0 is an accumulation point of $\Gamma - E$, then

$$C_D(f, z_0; L) \cap C_{\Gamma - E}(f, z_0) \neq \emptyset . \tag{9}$$

To prove this, let $\{\zeta_n\}$ be a sequence of points such that $\zeta_n \in \Gamma - E$ and $\zeta_n \to z_0$. Construct a simple closed curve γ_n passing through ζ_n such that γ_n surrounds z_0 and does not meet E. By a suitable choice of the sequence $\{\gamma_n\}$, we may assume that the diameter of γ_n converges to zero. Let z_n be the last point of intersection of L with γ_n. Then, it is obvious that the component, containing z_n, of the intersection of γ_n with D is a cross-cut of D whose end-points lie in $\Gamma - E$. From this fact follows that for every positive number r, $\overline{\mathfrak{D}}_r^*$ and M_r (defined before) have a point in common and hence (9) holds.

§ 2. Some classical theorems

We recall some important classical theorems which will be made use of for the sequel.

1. Let $w = f(z)$ be a single-valued meromorphic function in a domain D: $0 < |z - z_0| < r$ which has an essential singularity at z_0. Then, it is well-known that

(i) $C_D(f, z_0)$ *is the whole* w*-plane* (Weierstrass' theorem);

(ii) *the complement* $\mathscr{C} R_D(f, z_0)$ *of* $R_D(f, z_0)$ *with respect to the* w*-plane contains at most two points* (Picard's theorem);

(iii) $\mathscr{C} R_D(f, z_0) \subset A_D(f, z_0)$ (Theorem of Iversen [1]).

2. **Iversen's theorems**[1]. Let $w = f(z)$ be a non-rational meromorphic function in $|z| < \infty$ and $z = \varphi(w)$ be its inverse analytic function. Let c: $|w - \alpha| = r$ be an arbitrary circle in the w-plane. Suppose that $e(w, w_0)$ is an arbitrary (regular or algebraic) element of $z = \varphi(w)$ with center w_0 lying in (c): $|w - \alpha| < r$. Iversen [1] has proved that *it is possible to find a path* γ_w *inside* (c), *starting at* $w = w_0$ *and terminating at* $w = \alpha$, *such that there exists an analytic continuation of* $e(w, w_0)$ *of algebraic character*[2] *along* γ_w *except perhaps the end-point* $w = \alpha$ *of* γ_w; we call this property Iversen's *property* or (I)-*property*[3]. It is easy to prove that (I)-property is equivalent to the property that given any element $e(w, w_0)$ of $z = \varphi(w)$, an arbitrary curve Λ_w, starting from $w = w_0$ and ending at $w = w_1$, and an arbitrary strip[4] S containing Λ_w completely in its

[1] Nevanlinna [6], p. 291.

[2] Concerning notions of analytic continuation of algebraic character and (ordinary or essential) transcendental singularity, cf. Collingwood and Cartwright [1], pp. 99—103; Noshiro [4], pp. 43—73.

[3] Iversen's property of analytic functions has been systematically investigated by Stoïlow [1, 9].

[4] S denotes the union of all circular discs of constant radius and with center lying on Λ_w.

interior, we can find a path L_w, connecting w_0 and w_1, inside S, along which the analytic continuation of $e(w, w_0)$ is possible except perhaps at $w = w_1$.

Suppose that $w = f(z)$ has an asymptotic value α at $z = \infty$ along a curve L_z: $z = z(t)$, $0 \leqq t < 1$, $\lim_{t\to 1} z(t) = \infty$. Let $e_{z(t)}$ be the element of $z = \varphi(w)$ corresponding to $z(t)$. Then, the analytic continuation $\{e_{z(t)}, 0 \leqq t < 1\}$ of algebraic character along L_w: $w = w(t) = f(z(t))$, $0 \leqq t < 1$, $\lim_{t\to 1} w(t) = \alpha$ defines a transcendental (ordinary) singularity at $w = \alpha$. The converse is also true. If there exists an analytic continuation $\{e(w, w(t)), 0 \leqq t < 1\}$ along a path L_w: $w = w(t)$, $0 \leqq t < 1$, $\lim_{t\to 1} w(t) = \alpha$ which defines a transcendental singularity at $w = \alpha$, then $w = f(z)$ has an asymptotic value α at $z = \infty$ along the curve L_z: $z = z(t) = e(w(t), w(t))$, $0 \leqq t < 1$, $\lim_{t\to 1} z(t) = \infty$.

3. Gross' star theorem[1]. Let $w = f(z)$ be a non-rational meromorphic function and $z = \varphi(w)$ be its inverse. Let $e(w, w_0)$ be an arbitrary regular element of $z = \varphi(w)$. We continue analytically $e(w, w_0)$, using only regular elements, along every ray: $\arg(w - w_0) = \theta$ $(0 \leqq \theta < 2\pi)$ towards infinity. Then, there arise two cases whether the continuation defines a singularity ω_θ in a finite distance or not; in the former case, we call the ray a *singular* ray. For each singular ray: $\arg(w - w_0) = \theta$, we exclude the segment between the singularity ω_θ and $w = \infty$ from the w-plane. The remaining domain Δ_w is clearly a (simply connected) star domain in which the element $e(w, w_0)$ defines a (single-valued) regular branch of $z = \varphi(w)$. The star theorem of GROSS [1] states that *the set of θ of singular rays*: $\arg(w - w_0) = \theta$ $(0 \leqq \theta < 2\pi)$ *is of measure zero*; i. e., $e(w, w_0)$ can be continued (with rational character) to infinity along almost all rays from the center w_0 (Gross' *property*).

It is easy to show that Iversen's theorem is a direct consequence of Gross' theorem; i. e., Iversen's property follows from Gross' property. Gross' property is more metrical and less topological than Iversen's property.

As an application of the Gross star theorem, we prove that *if α is an exceptional value in the sense of* PICARD[2], *then α is an asymptotic value of $w = f(z)$ at $z = \infty$*. Without loss of generality, we may suppose that $\alpha = \infty$. Choose a point w_0 such that there exists an infinite number of elements $e_n(w, w_0)$ $(n = 1, 2, \ldots)$ of $z = \varphi(w)$ with center $w = w_0$. Then, by the Gross theorem, there exists at least one ray from $w = w_0$ along which every element $e_n(w, w_0)$ can be continued to infinity. Since there is only a

[1] NEVANLINNA [6], p. 292.

[2] This means that α is taken by $w = f(z)$ only finite times in $|z| < \infty$.

finite number of elements of $z = \varphi(w)$ with center $w = \infty$, the continuation of some $e_n(w, w_0)$ defines a transcendental singularity at $w = \infty$[1].

4. We now enunciate some fundamental theorems on meromorphic functions in the unit circle.

Theorem of FATOU[2]. *Let $w = f(z)$ be regular and bounded in the unit circle D: $|z| < 1$. Then, $w = f(z)$ has an angular limit $f(e^{i\theta})$ at almost every point $z = e^{i\theta}$ of Γ: $|z| = 1$.*

Theorem of F. and M. RIESZ[3]. *Let $w = f(z)$ be regular and bounded in the unit circle D: $|z| < 1$. If the boundary function $f(e^{i\theta})$ is equal to α on a subset of positive measure of Γ, then $f(z) \equiv \alpha$.*

NEVANLINNA[4] has extended these theorems to the case of meromorphic functions of bounded type.

Theorem of LINDELÖF-IVERSEN-GROSS[5]. *Let $w = f(z)$ be a function, meromorphic in the unit circle D: $|z| < 1$, which omits three different values. If $w = f(z)$ has an asymptotic value α along a simple curve L in D terminating at $z_0 = e^{i\theta_0}$, then $f(z)$ has necessarily the angular limit α at $z_0 = e^{i\theta_0}$.*

Theorem of KOEBE-GROSS[6]. *Let $w = f(z)$ be a function, meromorphic in $|z| < 1$, which omits three different values in $|z| < 1$, and let there exist two sequences $\{z_n^{(1)}\}$ and $\{z_n^{(2)}\}$ such that $|z_n^{(1)}| < 1$, $\lim_{n\to\infty} z_n^{(1)} = e^{i\theta_1}$; $|z_n^{(2)}| < 1$, $\lim_{n\to\infty} z_n^{(2)} = e^{i\theta_2}$ where $\theta_1 \neq \theta_2$. If there is a sequence of continuous curves γ_n joining $z_n^{(1)}$ to $z_n^{(2)}$ and contained in an annulus $1 - \varepsilon_n < |z| < 1$, where $\varepsilon_n > 0$, $\lim_{n\to\infty} \varepsilon_n = 0$, such that on γ_n we have $|f(z) - \alpha| < \eta_n$ where $\lim_{n\to\infty} \eta_n = 0$, then $f(z) \equiv \alpha$.*

II. Single-valued analytic functions in general domains

It belongs to one of the most important problems to study singularities, distribution of values, boundary-behaviours of analytic functions of a general domain of existence and their Riemann surfaces. In this chapter, we discuss mainly on single-valued analytic functions with a compact set of logarithmic capacity zero of essential singularities from the viewpoint of cluster sets[7].

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[1] Modifying the argument slightly, we see that this result also holds in the case where $f(z)$ is a single-valued meromorphic function in $0 < |z - z_0| < r$ with an essential singularity at $z = z_0$; i. e. (iii) holds.

[2] FATOU [1].

[3] F. and M. RIESZ [1].

[4] NEVANLINNA [6], p. 208 and p. 209.

[5] PHRAGMÉN-LINDELÖF [1], IVERSEN [1], GROSS [1].

[6] KOEBE [1, 2]; GROSS [1], pp. 35—36.

[7] Nevanlinna's theory of meromorphic functions (in the parabolic case) has been extended independently by G. AF HÄLLSTRÖM [2] and TSUJI [3, 7] to the case of a compact set of logarithmic capacity zero of essential singularities.

§ 1. Compact set of capacity zero and Evans-Selberg's theorem

1. We recall some basic properties of a compact set of capacity zero[1]. Let E be a bounded Borel set in the z-plane and μ be a non-negative completely additive set function defined for the Borel subsets of E. Then μ is called a positive mass-distribution on E. Let μ be a positive mass-distribution on E with total mass unity. Then

$$U^{\mu}(z) = \int_E \log \left| \frac{1}{z - \zeta} \right| d\mu(\zeta) \tag{1}$$

is called a (logarithmic) potential of distribution μ on E. Writing

$$V_{\mu}(E) = \sup_z U^{\mu}(z), \quad V = \inf_{\mu} V_{\mu}(E), \tag{2}$$

we define the (logarithmic) capacity $C(E)$ of E by

$$C(E) = e^{-V}.^{2} \tag{3}$$

Obviously $0 \leq C(E) < \infty$; if $E_1 \subset E_2$, then $C(E_1) \leq C(E_2)$; moreover, if there exists a sequence of bounded Borel sets E_n, such that $C(E_n) = 0$ for all n, and if $E = \bigcup_{n=1}^{\infty} E_n$ is bounded, then $C(E) = 0$.

2. Let us consider a domain D, containing $z = \infty$ in its interior, with boundary Γ. We suppose now that E is a compact set complementary to D. Let $\{D_n\}$ be an exhaustion of D such that each D_n is bounded by a finite number of simple closed analytic curves Γ_n and such that $\overline{D}_n \subset D_{n+1}$ $(n = 1, 2, \ldots)$. Denote by $g_n(z, \infty)$ Green's function of D_n with pole at $z = \infty$. Since $\{g_n(z, \infty)\}$ is a monotone increasing sequence, the limit is either a finite function $g(z, \infty)$ in D except for $z = \infty$ or a constant ∞. In the former case, $g(z, \infty)$ is called Green's function of D with pole $z = \infty$ and in the latter we say that there exists no Green's function of D. It is well-known that there exists no Green's function of D if and only if $C(E) = 0$[3].

3. Now, let D_0 be an arbitrary Jordan domain bounded by a simple closed analytic curve Γ_0 such that $\overline{D}_0 \subset D_1$. For simplicity, we put $G = D - \overline{D}_0$, $G_n = D_n - \overline{D}_0$. We denote by $\omega_n(z) = \omega(z, \Gamma_n, G_n)$ the harmonic measure with boundary values 0 on Γ_0 and 1 on Γ_n respectively. Since $\{\omega_n(z)\}$ is monotonically decreasing, this sequence converges uniformly on any compact set in G (Harnack's theorem); we denote the limiting function by

$$\omega(z) = \omega(z, \Gamma, G).$$

[1] Throughout this book, "capacity" always means "logarithmic capacity". Logarithmic capacity, logarithmic potential and harmonic measure are discussed in details in NEVANLINNA [6]. Concerning general potentials, cf. FROSTMANN [1], KAMETANI [4].

[2] In case $V = \infty$, we put $C(E) = 0$.

[3] NEVANLINNA [6], p. 123.

Evidently, $\omega(z)$ is harmonic on $G \cup \Gamma_0$[1]; $\omega(z) = 0$ on Γ_0 and $0 \leq \omega(z) < 1$ in G. By the minimum principle, if $\omega(z)$ vanishes at some point in G, then $\omega(z) \equiv 0$. If $\omega(z, \Gamma, G) \equiv 0$, then we say that E is of absolute harmonic measure zero (NEVANLINNA)[2]. If Γ contains a non-degenerate continuum, then $\omega(z, \Gamma, G) > 0$[3]. Accordingly, if Γ is of absolute harmonic measure zero, then Γ (and therefore E) is totally disconnected. Furthermore, Γ is of absolute harmonic measure zero if and only if $C(E) = 0$[4].

Remark. Letting z_i $(i = 1, 2, \ldots, n)$ vary on a compact set E, we denote by V_n the maximum value of the quantity

$$V(z_1, z_2, \ldots, z_n) = \prod_{k<\lambda}^{1\cdots n} |z_k - z_\lambda| .$$

Then, $\sqrt[\binom{n}{2}]{V_n}$ is monotonically decreasing and converges to a limit $\tau(E)$ which is named by FEKETE [1] the transfinite diameter of E. It is known that $C(E) = \tau(E)$[5].

4. We add a remark on metrical properties of a compact set of capacity zero[6]. Let E be a compact set. If for any positive number ε, we can cover E by a sequence of circular discs K_n of radius r_n such that $\Sigma r_n < \varepsilon$, then we say that E is of linear measure zero. Similarly, we define E to be of logarithmic measure zero, by replacing $\Sigma r_n < \varepsilon$ by $\Sigma (\log^+ 1/r_n)^{-1} < \varepsilon$. It is known that if E is of logarithmic measure zero, then $C(E) = 0$; if $C(E) = 0$, then E is of linear measure zero; their converses are not true.

5. Evans-Selberg's theorem. G. C. EVANS [1] and H. SELBERG [1] have proved independently the following

Theorem 1. *Let E be a compact set of capacity zero. Then there exists a positive mass-distribution μ on E with total mass unity, such that its potential*

$$u(z) = \int_E \log \left| \frac{1}{z - \zeta} \right| d\mu(\zeta) \tag{4}$$

is positively infinite at every point of E and at no other points.

Proof[7]. Given n points $a_1, a_2, \ldots, a_n$ on E, we form a polynomial $P(z) = (z - a_1)(z - a_2) \ldots (z - a_n)$. Denote by $\overline{M}_n$ the maximum modulus

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[1] Since $\omega_n(z) = 0$ on Γ_0, it follows, by Schwarz's principle of reflection, that $\omega(z)$ is also harmonic on Γ_0.

[2] The distinction whether $\omega(z, \Gamma, G)$ identically vanishes or not is independent of the choice of an exhaustion $\{D_n\}$ $(n = 0, 1, 2, \ldots)$ of D. Cf. NEVANLINNA [6], p. 119.

[3] NEVANLINNA [6], p. 120.

[4] NEVANLINNA [6], p. 126.

[5] NEVANLINNA [6], p. 135.

[6] Cf. NEVANLINNA [6], pp. 148—163; also KAMETANI [4].

[7] NOSHIRO [6], G. AF HÄLLSTRÖM [2]. This proof is essentially the same as that of EVANS [1], although Evans' original theorem is stated in the case of 3-dimensions.

of $P(z)$, letting z vary on E, i. e., $\overline{M}_n = \max_{z \in E} |P(z)|$, and by M_n the greatest lower bound of $\overline{M}_n$, letting n points $a_1, a_2, \ldots, a_n$ vary on E, i. e., $M_n = \inf \overline{M}_n$. Then, it is easily shown that M_n is the minimum of $\overline{M}_n$; in other words, by a suitable choice of $a_1^0, a_2^0, \ldots, a_n^0$ on E, there exists a polynomial

$$T_n(z) = (z - a_1^0)\,(z - a_2^0) \ldots (z - a_n^0)$$

with maximum modulus M_n. Remembering the definition of the transfinite diameter $\tau(E)$ of E and the relation $\tau(E) = C(E)$, we denote by V_n the maximum of

$$V(z_1, z_2, \ldots, z_n) = \prod_{k<\lambda}^{1 \cdots n} |z_k - z_\lambda| ,$$

letting z_i $(i = 1, 2, \ldots, n)$ vary on E. Let the maximum V_{n+1} be attained by $n + 1$ points $b_1, b_2, \ldots, b_{n+1}$ on E. From

$$\begin{aligned} V_{n+1} &= V(b_1, b_2, \ldots, b_{n+1}) \\ &= |(b_1 - b_2)\,(b_1 - b_3) \ldots (b_1 - b_{n+1})| \cdot V(b_2, b_3, \ldots, b_{n+1}) \end{aligned}$$

follows

$$|(b_1 - b_2)\,(b_1 - b_3) \ldots (b_1 - b_{n+1})| \geqq M_n , \tag{5}$$

for otherwise there would exist a point $b_1' \in E$ such that $V(b_1, b_2, \ldots, b_{n+1}) < V(b_1', b_2, \ldots, b_{n+1})$. By a cyclic change of suffices of b in (5), we have

$$V_{n+1} \geqq M_n^{\frac{n+1}{2}} \quad \text{and} \quad \sqrt[\binom{n+1}{2}]{V_{n+1}} \geqq \sqrt[n]{M_n},$$

whence follows

$$\lim_{n \to \infty} \sqrt[\binom{n+1}{2}]{V_{n+1}} = \lim_{n \to \infty} \sqrt[n]{M_n} = 0,$$

as $\tau(E) = C(E) = 0$.

Consider now the function

$$\begin{aligned} u_n(z) &= -\log \sqrt[n]{|T_n(z)|} \\ &= \frac{1}{n} \left(\log \left| \frac{1}{z - a_1^0} \right| + \log \left| \frac{1}{z - a_2^0} \right| + \cdots + \log \left| \frac{1}{z - a_n^0} \right| \right) ; \end{aligned}$$

$u_n(z)$ is clearly a potential defined by a certain distribution of equal point masses on E with total mass unity and for every point z on E, $u_n(z) \geqq m_n$ where $m_n = -\log \sqrt[n]{M_n}$. Since $m_n \to \infty$, we can find a sequence of integers $\{n_j\}$ such that $m_{n_j} \geqq 2^j$ $(j = 1, 2, \ldots)$. Put $U_j(z) = 2^{-j} u_{n_j}(z)$ $(j = 1, 2, \ldots)$. Then, $U_j(z)$ is a potential of distribution of equal point masses on E with total mass 2^{-j} and evidently $U_j(z) \geqq 1$ on E. Consider finally the function

$$u(z) = \sum_{j=1}^{\infty} U_j(z) = \lim_{\nu \to \infty} \sum_{j=1}^{\nu} U_j(z) .$$

Then, $u(z)$ is a required potential. In fact, it is a potential of positive mass-distribution on E with total mass unity and hence of the form (4). At every point z of E, $u(z) = +\infty$ as $u(z) \geqq \sum_{j=1}^{\nu} U_j(z) \geqq \nu$ for all ν. If $z \in \mathscr{C}E$ and if z has a distance ϱ from E, then clearly $u(z) \leqq \log 1/\varrho$.

Remark. For convenience, we shall call the potential $u(z)$, in Theorem 1, an Evans-Selberg's *potential*. For a given compact set of capacity zero, Evans-Selberg's potential is not unique (G. AF HÄLLSTRÖM [3]).

6. For the sequel, it will be convenient to state some properties of Evans-Selberg's potential $u(z)$. Clearly $u(z)$ is harmonic outside E except for $z = \infty$ and its boundary value at every point of E is $+\infty$. In the neighborhood of $z = \infty$, $u(z)$ is of the form

$$u(z) = -\log|z| - \omega(z), \tag{6}$$

where $\omega(z) = \int_E \log|1 - \zeta/z| d\mu(\zeta)$ is harmonic at $z = \infty$. Let $v(z)$ be its conjugate harmonic function and put

$$w(z) = u(z) + iv(z). \tag{7}$$

Then the function $w(z)$ is many-valued and regular outside E except for $z = \infty$, the infinity being a logarithmic singularity. However the derivative $w'(z) = u_x(z) - iu_y(z)$ is obviously single-valued and regular throughout the domain $\mathscr{C}E$, $z = \infty$ being a simple zero-point of $w'(z)$, and has a singularity at every point of E. Consequently, the many-valuedness of $w(z)$ arises only in its imaginary part by some additive constants. It is easy to show that the level curve Γ_λ: $u(z) = \lambda$ $(-\infty < \lambda < \infty)$ consists of a finite number of simple closed curves surrounding E, by the minimum principle of harmonic function, and that the function $\lambda - u(z)$ is no other than Green's function $g(z, \infty)$ in the exterior of Γ_λ. Thus we see that if there are p closed curves of Γ_λ, then $w'(z)$ has $p - 1$ finite zero-points in the exterior of Γ_λ and moreover that

$$\int_{\Gamma_\lambda} dv(z) = \int_{\Gamma_\lambda} \frac{\partial u}{\partial n} ds = 2\pi, \tag{8}$$

where ds denotes the arc length and n the inner normal (see HÄLLSTRÖM [2], pp. 14—17).

Remark. Recently extensions of the Evans-Selberg theorem and related theorems have been obtained by RUDIN [1], UGAERI [1], HONG [1] and INOUE [1].

§ 2. Meromorphic functions with a compact set of essential singularities of capacity zero

1. At the beginning, we prove

Theorem 1. *Let E be a compact set of capacity zero and D be a domain containing E in its interior. Suppose that $w = f(z)$ is a single-valued mero-*

morphic function in $D - E$ and has a transcendental singularity at every point z_0 of E. Then, the cluster set $C_{D-E}(f, z_0)$ of $f(z)$ at $z = z_0$ is the whole w-plane (Nevanlinna [6]).

Proof. Obviously we have only to prove that $f(z) = u(z) + iv(z)$ is not bounded in any neighborhood of every point z_0 of E. Otherwise, $f(z)$ would be bounded in the intersection of $D - E$ with a circular disc (c): $|z - z_0| < r$. Describe a simple closed curve Γ, surrounding z_0, in $(D - E) \cap (c)$[1] and denote by Δ the remaining domain obtained by excluding E from the interior of Γ. Let $\bar{u}(z)$ be the harmonic function in the interior of Γ, such that $\bar{u}(z) = u(z)$ on Γ, and $u^*(z)$ be the Evans-Selberg potential which may be supposed to be positive in (c). Consider $U(z) = u(z) - \bar{u}(z) - \varepsilon u^*(z)$ in Δ for any positive number ε. Then, clearly $U(z) \leqq 0$ in Δ; hence $u(z) \leqq \bar{u}(z)$ in Δ. Similarly $\bar{u}(z) \leqq u(z)$ in Δ. Thus $u(z) = \bar{u}(z)$ in Δ. Accordingly z_0 is a removable singularity for $f(z)$; this is a contradiction.

Remark. In the proof, we have used only the fact that if a harmonic function is bounded in a neighborhood of a compact set of capacity zero, this set is removable for the harmonic function. Obviously Theorem 1 remains valid if E is a Painlevé *null-set*[2], i. e. if E consists of *A B removable points*.

Theorem 2. *Let E be a compact set of capacity zero contained in a domain D. Suppose that $w = f(z)$ is a single-valued meromorphic function in $D - E$ which has an essential singularity at every point z_0 of E. Then, $w = f(z)$ assumes every value infinitely often in any neighborhood of z_0 with a possible exceptional set of values of capacity zero; i. e. $\mathscr{C}R_{D-E}(f, z_0)$ is at most of capacity zero* (G. af Hällström [2], Kametani [2]).

Proof. By Theorem 1, $R_{D-E}(f, z_0)$ is everywhere dense in the w-plane. Without loss of generality, we may suppose that $w = \infty$ belongs to $R_{D-E}(f, z_0)$. Let r be any positive number and (c) be a circular disc $|z - z_0| < r$. Describe a simple closed curve Γ, surrounding z_0, in $(D - E) \cap (c)$. Denote by Δ_r the domain $(\Gamma) - E$, where (Γ) is the interior of Γ, and by $\mathfrak{D}_r$ the value set of $f(z)$ in Δ_r. It is easily shown that the compact set $\mathscr{C}\mathfrak{D}_r$, which is complementary to $\mathfrak{D}_r$ with respect to the w-plane, does not contain any non-degenerate continuum; i. e. $\mathscr{C}\mathfrak{D}_r$ is totally disconnected. We show that $\mathscr{C}\mathfrak{D}_r$ is of capacity zero. Otherwise, there would exist a non-constant bounded harmonic function $U(w)$ in $\mathfrak{D}_r$. We

[1] As E is of capacity zero, E is of linear measure zero. Consequently we can adopt as Γ a circumference $|z - z_0| = \varrho$ for almost every positive number $\varrho < r$. But our selection of Γ depends upon only the property that E is totally disconnected.

[2] If there exists no non-constant single-valued bounded analytic function in the exterior of a totally disconnected compact set E, then E is called a Painlevé null-set or said to consist of *A B* removable points. It is easily proved that if E is of linear measure zero, then E is a Painlevé null-set (Ahlfors [3], Ahlfors-Beurling [1]). For related theorems, cf. A. S. Besicovitch [1], Cartwright [3].