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Recursive Estimation and Time-Series Analysis

An Introduction

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With 54 Figures

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To
Wendy, Timothy, Melanie and Jeremy

Preface

This book has grown out of a set of lecture notes prepared originally for a NATO Summer School on *"The Theory and Practice of Systems Modelling and Identification"* held between the 17th and 28th July, 1972 at the Ecole Nationale Superieure de L'Aeronautique et de L'Espace. Since this time I have given similar lecture courses in the Control Division of the Engineering Department, University of Cambridge; Department of Mechanical Engineering, University of Western Australia; the University of Ghent, Belgium (during the time I held the IBM Visiting Chair in Simulation for the month of January, 1980), the Australian National University, and the Agricultural University, Wageningen, the Netherlands. As a result, I am grateful to all the recipients of these lecture courses for their help in refining the book to its present form; it is still far from perfect but I hope that it will help the student to become acquainted with the interesting and practically useful concept of recursive estimation. Furthermore, I hope it will stimulate the reader to further study the theoretical aspects of the subject, which are not dealt with in detail in the present text. The book is primarily intended to provide an introductory set of lecture notes on the subject of recursive estimation to undergraduate/Masters students. However, the book can also be considered as a "theoretical background" handbook for use with the CAPTAIN Computer Package. This 'Computer Aided Program for Time Series Analysis and the Identification of Noisy Systems' was originally conceived by the author in the mid nineteen sixties and was developed initially in the Control Division of the Engineering Department at the University of Cambridge in 1971 on the basis of recursive algorithms developed during the previous six years (Young *et al.*, 1971; Shellswell, 1971). A *command mode* version of the package was developed as an alternative to the original interactive, *conversational mode* version, by the Institute of Hydrology in England (Venn and Day, 1977) and in this form, it has been acquired by the Commonwealth Scientific and Industrial Research Organisation (CSIRO) of Australia for use in its CSIRONET nationwide computing system (Freeman, 1981). More recently, advanced versions of CAPTAIN, based on some of the more sophisticated procedures discussed in the present book, have been developed by the author and his colleagues at the Centre for Resource and Environmental Sciences, (CRES), ANU, for use in the UNIVAC 1100 series computer. A more comprehensive, "user friendly" version of the package is currently under development by the author and Mr. John Hampton in the Department of Environmental Sciences, University of Lancaster, Lancaster, LA1 4YQ, England. This version will be available initially for use on the VAX 11/780 computer. Also a microcomputer version MICROCAPTAIN has recently been developed by the author (see Epilogue) for use on the APPLE II and EPSON HX-20 microcomputers. Any enquiries about any of these CAPTAIN programs should be addressed to me at the above address.

Acknowledgements

The final appearance of this book has been delayed for various reasons. Foremost amongst these are the various changes in location of my family and myself during the preparation of the text. It was begun in 1979, while I was spending a period of study leave at my old College, Clare Hall, in Cambridge; it was continued, very much as a spare time activity between numerous environmental projects, at the Centre for Resource and Environmental Studies, Australian National University, Canberra, between 1980 and 1981; and it has been completed, whenever time has allowed, at the Department of Environmental Science, University of Lancaster, between 1981 and 1983.

The overlong gestation period has meant that, at various times, a number of people have read draft chapters of the book and suggested improvements. In particular, I would like to thank Dr. Paul Steele, Dr. Tony Jakeman, and Miss Christina Sirakoff for their help and encouragement. Mrs. June Harries (Canberra) and Mrs. Glenys Booth (Lancaster) have typed most of the manuscript, often from poorly handwritten drafts, and I am extremely grateful to both of them for their patience and the quality of their work. A number of the diagrams were drawn by Elizabeth Barta.

Of course all the errors, omissions and other deficiencies of the book are my sole responsibility. Like most authors, if I was writing the book again, I would change many aspects of the presentation. Having gone through the not inconsiderable labours of preparing a "camera-ready" manuscript, however, I find such a possibility daunting in the extreme. I hope, therefore, the reader will not be too disappointed in the final outcome.

Finally, in dedicating this book to my wife and family, I am acknowledging their unfailing support, without which the book would never have appeared.

Peter Young
Lancaster, England.
January 1984.

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Algorithms

1 Introduction

*Nous traiterous particulièrement le
problème suivant, tant à cause de son utilite
pratique que de la simplicité de la solution*

K.F. Gauss
Méthode des Moindres Carrés
Translation by J. Bertrand, 1855.

The concept of least squares estimation is inextricably linked with the name of Karl Friedrick Gauss. Although Legendre was responsible for the first published account of the theory in 1805 and, indeed, first coined the term "least squares", it was Gauss who developed the method into a statistical tool, embedding it within a statistical framework involving a probabilistic treatment of observational errors.

Gauss's first published exposition on least squares appeared in his famous *Theoria Motus Corporum Coelestum* which appeared in 1809 when he was 31 years of age. But, as D.A. Sprott (1978) has pointed out recently in his excellent review of Gauss's contributions to statistics, the basic ideas were most probably formulated while he was still in his twenties.

In the *Theoria Motus*, the discussion on least squares appears in relation to an important practical problem; namely the estimation of the six constant coefficients or "parameters" that determine the elliptical orbit of a planetary body, on the basis of $n > 6$ observations. His second exposition on the subject was presented in a series of papers (1821, 1823, 1826) which were collected together under the title *Theoria Combinationis Erroribus Minimum Obnoxiae*. Here he abandoned the previous "inferential" treatment delineated in the *Theoria Motus* and concentrated on a "decision theoretic" approach, in which he restricted attention to estimates that are a linear function of the observations. And it was here that he presented perhaps his most famous statistical theorem that, *Among all linear error-consistent estimates, the least squares estimate has minimum mean square error*.

But our current interest lies elsewhere in the *Theoria Combinationis:* in a quite short and apparently little known section of some five pages (Bertrand, 1855, p. 53-58; Trotter, 1957, p. 67-72) Gauss shows how it is possible *to find the changes which the most likely values of the unknowns undergo when a new equation (observation) is adjoined and to determine the weights of these new determinations*. In other words, and to use more contemporary terminology, he developed an algorithm for sequentially or *recursively* updating the least squares parameter estimates on receipt of additional data.

Bertrand's French translation of Gauss's work appeared in 1855 under the appropriate title "Méthode des Moindres Carrés" and was authorised by Gauss himself.

This translation, together with a commentary provided by the present author, appears in Appendix 2. In addition to its importance in historical terms, Gauss's analysis is interesting because it demonstrates the elegance of his approach and the power of his mind: without the advantages of matrix algebra which, as we shall see, considerably simplify the derivation, Gauss was able to obtain the recursive least squares algorithm with consummate ease.

Gauss's analysis represents the birth of recursive least squares theory; a theory so much ahead of its time that it would lie dormant for almost a century and a half before it was rediscovered on two separate occasions: first by the statistician R.L. Plackett in 1950; and then later and in a more sophisticated form, as the core of the linear filtering and prediction theory evolved by the control and systems theorist R.E. Kalman (1960).

Not surprisingly, perhaps, Plackett's paper went almost unnoticed in the pre-computer age of the early nineteen fifties. Harking back to Gauss, he re-worked the original results in more elegant vector-matrix terms and developed an algorithm for the general case in which additional observations occur in sets $S > 1$. In the present book, like Gauss, we restrict the analysis largely to $S = 1$, although the extension to $S > 1$ is straightforward and is discussed in certain special cases.

Kalman's results, almost certainly obtained without knowledge of either Gauss's or Plackett's prior contributions, were developed within the context of state variable estimation and filter theory, using an argument based on orthogonal projection. Not only were Kalman's results mathematically elegant in providing a computationally straightforward solution to the optimal filtering problem, which had a number of advantages over the earlier Wiener solution (Wiener, 1949), but they also had good potential for practical application. Not surprisingly, therefore, they caused quite a revolution in the automatic control and systems field providing, during the next ten years, a rich source of research material for control and systems analysts. Subsequently, the term *Kalman Filter* has become widely used, not only in the academic and industrial world of automatic control but also in other disciplines such as statistics and economics.

It is now well known that the Kalman filter estimation algorithm can be derived in various ways; via orthogonal projection, as in Kalman's exposition; as well as from the standpoint of maximum likelihood or Bayesian estimation. It can also be developed in various different forms, for application to both discrete (Kalman, 1960) and continuous (Kalman and Bucy, 1961) time-series. But,in all its forms, it has had a profound effect on data processing during the last two decades, being used in applications ranging from trajectory and orbit estimation to the forecasting of economic time-series.

Sprott (1978) has questioned whether the Kalman filter is really a significant 'development' of the Gauss-Plackett recursive least squares algorithm. While it is true that the Gauss-Plackett recursion formulae are an essential aspect of the

Kalman filter equations, it is also clear, as we shall see in this book, that Kalman considerably extended the theory both to allow for the estimation of *time-variable* parameters or *states*, and to handle the analysis of statistically *non-stationary* time-series. Nevertheless the Gauss-Plackett recursion is undoubtedly the central component of the Kalman filter and the basis of most other recursive least squares algorithms. Thus a good understanding of its function in a data processing sense is an essential pre-requisite for the practical application of the algorithm. It is the provision of such understanding, therefore, which is one of the primary aims of this book.

As Gauss pointed out so succinctly in the quotation at the beginning of this chapter, recursive least squares theory is both simple and useful. Here we will exploit this simplicity and take the reader gently through the mysteries of the subject, avoiding wherever possible undue rigour and complexity. In the spirit of Gauss, we will concentrate on mathematical analysis which, while it is often algebraic in form, also has sufficient statistical content to ensure that the reader is fully aware of the important statistical aspects of the results. We will, however, allow ourselves one luxury not available to Gauss and simplify the analysis still further by resort to matrix algebra, assuming that the reader is already acquainted with such analysis; has access to a good text on the subject; or finds that the background notes in Appendix 1 of the book provide sufficient revision.

Finally, to emphasize the practical utility of the various recursive least squares algorithms which will emerge during the analysis, we will provide a number of simulation and practical examples, with applications which range from the man-made world of engineering to the more natural world of ecology and the environment. Many other applications in diverse areas, from economics to hydrodynamics, are discussed in a variety of technical papers produced by the author and his colleagues over the past few years and these are either referred to in the text or listed in the bibliography. Some more recent references are discussed in a short Epilogue at the end of the book.

The text is divided into two major parts: the first is primarily concerned with the estimation of constant or time-variable parameters in general models which are *linear-in-the-parameters;* the second shows how the procedures developed in the first part can be modified to handle the analysis of stochastic time-series and so provide algorithms for the recursive estimation of parameters and states in stochastic dynamic systems. In sympathy with the introductory nature of this book, there has been a conscious attempt to simplify the mathematical analysis as much as possible, particularly in the early chapters, so as to enhance the readability of the book and avoid an overly esoteric presentation. For the reader unfamiliar with some of the mathematics used in the book, Appendix 1 provides background notes, not only on matrix algebra but also on probability and statistics, as well as some very simple concepts in dynamic systems. In all cases, the results in Appendix 1 are chosen