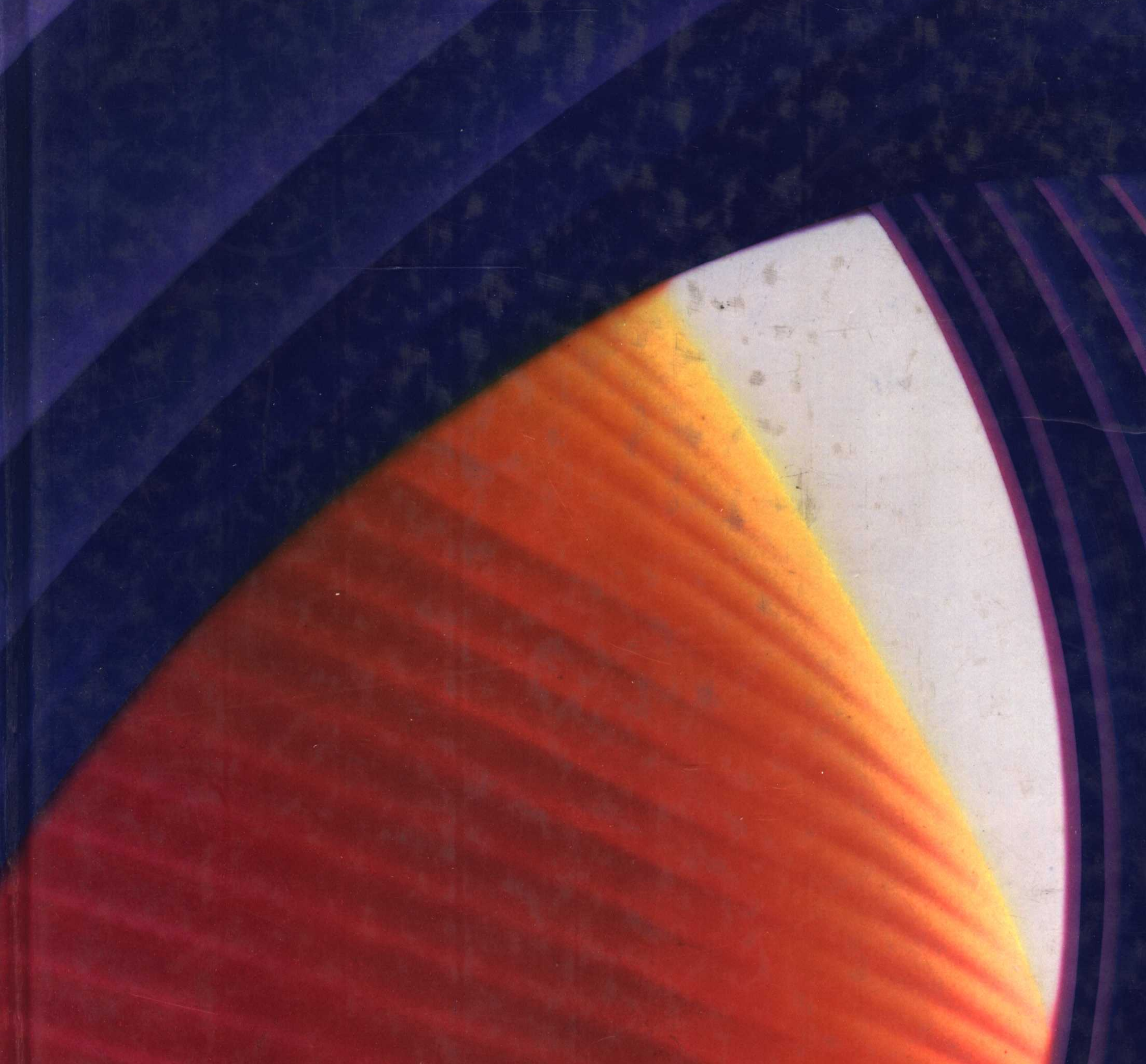
APPLIED CALCULUS



JOHN C. HEGARTY

APPLIED CALCULUS

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BENTLEY COLLEGE

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PREFACE

Applied Calculus has been written for use in an introductory calculus course for students pursuing professional careers in the management, life, or social sciences. The goal of the book is to present, in an intuitive and clear manner, the fundamental concepts of differential and integral calculus to a student who is primarily interested in learning how calculus can be used. For this reason numerous examples and applications, where appropriate, are given in each section to illustrate and enhance new concepts. In addition, attention has been paid to making the explanations and examples as complete and detailed as possible to increase the readability of the book.

Although the presentation is not mathematically rigorous, basic concepts such as the limit, continuity, the derivative, and definite and indefinite integrals are developed carefully and systematically. Because students assimilate mathematical concepts more quickly and easily when they can visualize their meaning, there is a heavy reliance and emphasis on graphing not only in the presentation of new material, but also in the examples and in the problems. For example, graphs are used in Exercises 3.1, 3.2, and 3.3 to test and enhance a student's understanding of the derivative and the rules of differentiation. They are also used in Exercises 4.1 and 4.3 to test and enhance a student's understanding of the meaning and significance of the signs of the first and second derivatives. Exercises of this type have the advantage that no lengthy algebraic analysis is required; as a result a student can attain a considerable understanding of the basic concepts in a short period of time.

Basic curve sketching, whenever appropriate, is emphasized. Although curve sketching can, in many cases, be carried out more quickly with a computer or a calculator, critical points and inflection points are of such fundamental importance that a student should know how to determine and, simultaneously, to *visualize* the behavior of a function in the vicinity of each type. The ability to visualize the results of an analysis should be an important objective of any calculus course; curve sketching provides such a vehicle. For this reason the development of the curve-sketching process is carried out in detail for each example in Section 4.4. The same attention to detail is used in the curve-sketching examples for exponential and logarithmic functions in Sections 5.3 and 5.4.

The number and scope of the applications have been kept at a reasonable level. Students should realize that calculus is an important analytical tool in many disciplines and should gain some experience with its applications. On the other hand, there may be a price to pay when students are confronted with an overwhelming number of applications. Students often spend an inordinate amount of time struggling to understand the dynamics of an application without gaining a commensurate increase in their understanding of the fundamental concepts of calculus. Since the majority of students enrolled in applied calculus courses are in the fields of business and economics, the majority of applications are in these areas. The book contains two new elementary business applications developed by the author. Both have recently been published.*

Through problems in the exercises, I have attempted, in a small way, to focus each student's attention on the relationship between differential and integral calculus and on the types of problems that each addresses. The problems range from simple true–false questions in Exercise 6.1 to water supply and consumers' surplus problems in Exercises 6.3 and 6.4; the latter require a student to use optimization techniques to find the upper limit for a definite integral which is to be evaluated.

The text also makes extensive use of tables as visual aids or summaries. Tables are used extensively in Chapter 2 in developing the concept of the limit. Tables that highlight the signs of the first and second derivatives are used throughout Chapter 4. Table 1 of Section 3.3 describes the relationship between the last operation in an algebraic function and the appropriate rule used to differentiate the function. Table 1 in Section 6.1 lists the names of derivatives and their antiderivatives in some basic applications.

Supplements

An Instructor's Manual which contains

- complete solutions to all exercises and review problems,
- a syllabus for one- and two-semester courses,
- a listing of business applications by course,
- a review of algebra, which an instructor can reproduce for students who may be weak in the mathematical prerequisites.

A Student Study Guide and Solutions Manual which contains

- terms, concepts, and formulas,
- over 80 worked-out problems with solutions tips,
- solutions to most problems with emphasis on word problems,
- a review of algebra.

*1. Hegarty, J. C. "A Depreciation Model for Calculus Classes," *The College Mathematics Journal*, May 1987, Vol. 18, No. 3, pp. 219–221.

2. Hegarty, J. C."Calculus Model of Sum-of-the-Years-Digits Depreciation," *Mathematics and Computer Education*, Fall 1987, Vol. 21, No. 3, pp. 159–161.

Three Test System options averaging 50 questions per chapter

- Printed
- IBM PC compatible
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John C. Hegarty

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CONTENTS

1 PRELIMINARIES 1

1.1 FUNCTIONS AND THEIR GRAPHS 2
1.2 LINEAR FUNCTIONS 11
1.3 APPLICATIONS OF LINEAR FUNCTIONS 25
1.4 QUADRATIC FUNCTIONS AND APPLICATIONS 34
1.5 OTHER TYPES OF FUNCTIONS 44
REVIEW PROBLEMS 53

2 THE DERIVATIVE 57

2.1 LIMITS 58
2.2 CONTINUITY 73
2.3 AVERAGE RATE OF CHANGE 82
2.4 DERIVATIVE OF A FUNCTION 91
REVIEW PROBLEMS 103

3 DIFFERENTIATION AND APPLICATIONS 107

3.1 POWER RULE; SUM RULE 108
3.2 PRODUCT AND QUOTIENT RULES 118
3.3 GENERAL POWER RULE; CHAIN RULE 129
3.4 MARGINAL ANALYSIS 137
3.5 HIGHER-ORDER DERIVATIVES 144
3.6 IMPLICIT DIFFERENTIATION; RELATED RATES 147
REVIEW PROBLEMS 159

4 OPTIMIZATION AND CURVE SKETCHING 163

4.1 MAXIMA AND MINIMA 164
4.2 OPTIMIZATION 180
4.3 SECOND-DERIVATIVE TEST; INFLECTION POINTS 194

4.4 CURVE SKETCHING 204
4.5 THE DIFFERENTIAL 216
REVIEW PROBLEMS 226

5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS 231

5.1 EXPONENTIAL FUNCTIONS 232
5.2 LOGARITHMIC FUNCTIONS 246
5.3 DERIVATIVES OF EXPONENTIAL FUNCTIONS 257
5.4 DERIVATIVES OF LOGARITHMIC FUNCTIONS 265
5.5 APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS 274
5.6 NEWTON'S METHOD 283
REVIEW PROBLEMS 294

6 INTEGRATION 297

6.1 ANTIDERIVATIVES 298
6.2 AREA UNDER A CURVE 311
6.3 DEFINITE INTEGRAL 323
6.4 AREA BETWEEN TWO CURVES 340
6.5 ADDITIONAL APPLICATIONS OF THE DEFINITE INTEGRAL 350
REVIEW PROBLEMS 360

7 TECHNIQUES OF INTEGRATION 365

7.1 INTEGRATION BY SUBSTITUTION 366
7.2 INTEGRATION BY PARTS 373
7.3 TABLES OF INTEGRALS 380
7.4 NUMERICAL INTEGRATION 384
7.5 IMPROPER INTEGRALS 394
7.6 L'HÔPITAL'S RULE 401
REVIEW PROBLEMS 407

8 MULTIVARIABLE CALCULUS 411

8.1 FUNCTIONS OF TWO VARIABLES 412
8.2 PARTIAL DERIVATIVES 425
8.3 MAXIMA AND MINIMA 434
8.4 LAGRANGE MULTIPLIERS 442

8.5 LEAST SQUARES METHOD 450
8.6 TOTAL DIFFERENTIAL 460
8.7 DOUBLE INTEGRALS 467
8.8 MORE DOUBLE INTEGRALS 474
REVIEW PROBLEMS 485

9 DIFFERENTIAL EQUATIONS **489**

9.1 SOLUTIONS OF DIFFERENTIAL EQUATIONS 490
9.2 SEPARATION OF VARIABLES 497
9.3 EXACT EQUATIONS 503
9.4 APPLICATIONS 510
REVIEW PROBLEMS 522

10 PROBABILITY AND CALCULUS **525**

10.1 DISCRETE RANDOM VARIABLES 526
10.2 CONTINUOUS RANDOM VARIABLES 536
10.3 EXPECTED VALUE AND VARIANCE 545
REVIEW PROBLEMS 553

TABLES **557**

A Values of $(1 + i)^n$ 558
B e^x and e^{-x} 559
C Natural Logarithms 560
D Table of Integrals 561

ANSWERS TO ODD-NUMBERED PROBLEMS **563**

INDEX **603**

1.1 Functions and Their Graphs
1.2 Linear Functions
1.3 Applications of Linear Functions
1.4 Quadratic Functions and Applications
1.5 Other Types of Functions

1

PRELIMINARIES

INTRODUCTION

Calculus is the branch of mathematics that enables us to analyze or determine the effects of change. For example, the president of a fast-food chain would like to know what effect a 25¢ increase in the price of the company's popular cheeseburger will have on sales. Similarly, an ecologist would want to know how much the acidity of a lake will be changed over the next ten years by the acid content of the rainfall during this period.

In order to analyze the effect that a change in one variable has on another, it is necessary to know the mathematical relationship between the two variables. This relationship is generally expressed as a function. This chapter is devoted to studying some simple functions, their graphs, and applications.

1.1 FUNCTIONS AND THEIR GRAPHS

When the value of one variable depends on the value of a second variable, we say that the first is a function of the second and attempt to express the relationship mathematically. For example, A, the area of a square, depends on x, the length of each side, and the relationship between the two variables can be written as

$$A = x^2$$

The variable A is also said to be a *function* of the variable x. For each value that is assigned to x, the equation enables us to find the corresponding value of A; for example, if $x = 5$, then $A = (5)^2 = 25$. The variable x to which values are assigned is called the **independent** variable; A, the variable whose value is found from the equation, is called the **dependent** variable. The functional relationship between two variables is expressed in the following definition.

Definition A **function** f is a rule that matches or pairs each value of an independent variable, say, x, with exactly one value of a second variable, say, y. Each matching is written as an **ordered pair** (x, y) and y is said to be a function of x.

For example, the function f defined by the equation

$$y = x^2 + 1$$

is a rule that tells us to square each value of x and then to add 1 to get the corresponding value of y. Thus the number $x = 2$ is paired with the number

$$y = (2)^2 + 1 = 5$$

written as $(2, 5)$.

In applications, variables other than x and y are often used in representing functions, as illustrated in Example 1.

Example 1 The Speedy Rent-a-Car Agency charges \$15 per day plus 25¢ per mile for its compact autos. Find an equation that gives D, the charge in dollars, as a function of M, the daily mileage.

Solution The equation for D contains two terms: (1) the \$15 that is paid to take the car onto the road and (2) the mileage charge, in dollars, which is calculated by multiplying M by 0.25. Adding these two terms gives us the equation

$$D = 15 + 0.25M$$

If we drive 200 miles during the day, the charge becomes

$$D = 15 + 0.25(200) = \$65$$ ■

$f(x)$ Notation

When we define a function, we often write the variable y as $f(x)$ (read "f of x"). This notation is very useful in calculus because any value assigned to x is retained in the parentheses while we determine the corresponding value of y. For example, suppose the function f is defined as

$$y = f(x) = x^2 + 2x - 1$$

If x is assigned the value 3, the corresponding value of y, written $f(3)$, becomes

$$\begin{aligned} f(3) &= (3)^2 + 2(3) - 1 = 9 + 6 - 1 \\ &= 14 \end{aligned}$$

This tells us that $x = 3$ is paired with $y = 14$.

Example 2 Suppose the function f is defined by the equation

$$f(x) = 3x^2 - 5x + 7$$

Find $f(4)$ and $f(-1)$.

Solution We find $f(4)$ by substituting 4 for x everywhere in the equation that defines $f(x)$:

$$\begin{aligned} f(4) &= 3(4)^2 - 5(4) + 7 \\ &= 35 \end{aligned}$$

Similarly, we get for $f(-1)$

$$f(-1) = 3(-1)^2 - 5(-1) + 7 = 15$$ ■

In addition to $f(x)$, functional notation can be described by other letters such as $g(x)$ and $A(x)$, which may be better descriptors of the relationship between the independent and dependent variables. This is particularly true in applications.

Example 3 A pipe on an oil rig has ruptured and crude oil is spilling into the sea in all directions, forming a circular oil slick. The radius of the oil slick is a function of t, the time, in hours, from the rupture; the relationship is given by the equation

$$R(t) = 1000\sqrt{t}$$

where $R(t)$ is the radius, in feet. An additional crew has been dispatched to cap the well and to make the necessary repairs; however, it will take 36 hours to stop the leaking.

(a) How large will the radius of the oil slick be when the pipe is repaired?

(b) How large an area will the oil slick cover?

Solution (a) The radius of the oil slick is found by substituting 36 for t into the equation, yielding

$$\begin{aligned} R(36) &= 1000\sqrt{36} \\ &= 1000(6) = 6000 \text{ ft} \end{aligned}$$

(b) Recall that the area of a circle is given by the formula

$$A = \pi R^2$$

We get for the area

$$A = \pi(6000)^2 = 36{,}000{,}000\pi \text{ ft}^2$$

This information is useful to the owners of the rig because it enables them to estimate the cost of cleaning up the oil spill. ■

The $f(x)$ notation is particularly useful when x is replaced by an algebraic expression instead of a number. This occurs often when analyzing a function in calculus.

Example 4 If the function f is defined as

$$f(x) = x^2 - 3x + 2$$

find $f(1 + h)$.

Solution We can find $f(1 + h)$ by substituting $1 + h$ for x everywhere in the equation

$$f(1 + h) = (1 + h)^2 - 3(1 + h) + 2$$

The expression on the right-hand side can be simplified by expanding and grouping like terms together:

$$\begin{aligned} f(1 + h) &= (1 + 2h + h^2) - 3 - 3h + 2 \\ &= 1 + 2h + h^2 - 3 - 3h + 2 \\ &= h^2 - h \end{aligned}$$

This result tells us that when x is equal to $1 + h$, the corresponding value of y is $h^2 - h$. ■

Domain

When we work with a function, it is necessary to know what values can be assigned to the independent variable; these values constitute the **domain** of the function.

In Example 1 (Speedy Rent-a-Car), the variable M has meaning only for nonnegative values. Therefore, we say that the domain consists of the values of M for which $M \geq 0$.

When the domain is not given or cannot be inferred from the context of the problem, the domain consists of all real values of x for which the function is defined.

Example 5 Find the domain of each of the following functions:

(a) $f(x) = x^2 - 2$ (b) $f(x) = \dfrac{3}{2 - x}$ (c) $f(x) = \sqrt{x - 1}$

Solution (a) The expression $x^2 - 2$ produces a real number for every value of x that is substituted into it, so we conclude that the domain consists of all real numbers.

(b) The quantity $\dfrac{3}{2 - x}$ produces a real number for every value of x except $x = 2$, where division by zero occurs; so the domain consists of all real numbers except $x = 2$.

(c) The quantity $\sqrt{x - 1}$ generates real numbers only when $x - 1 \geq 0$, so the domain consists of values of x that are greater than or equal to 1, that is, $x \geq 1$. ■

Graph of a Function

When we work with a function, it is helpful to visualize the relationship between the independent variable x and the dependent variable $y = f(x)$. We accomplish this by sketching the *graph* of the function.

Recall that an ordered pair (x, y) or $(x, f(x))$ can be represented geometrically as a point in a Cartesian or xy-coordinate system; the point representing the ordered pair (3, 2) is shown in Figure 1. The **graph** of a function f is the set of all points corresponding to the ordered pairs $(x, f(x))$ that satisfy the equation defining the function.

When a function is defined by a simple equation, its graph can usually be sketched by finding a representative group of points and then connecting them by means of a smooth curve, as the next two examples illustrate.

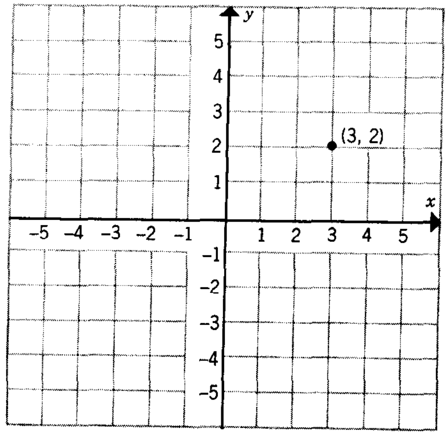

Figure 1

Example 6 Sketch the graph of the function $f(x) = x^2 - 2$.

Solution A table containing representative values of x and y is used to generate points to sketch the graph in Figure 2.

x	$y = x^2 - 2$
−3.0	7.0
−2.0	2.0
−1.0	−1.0
0.0	−2.0
1.0	−1.0
2.0	2.0
3.0	7.0

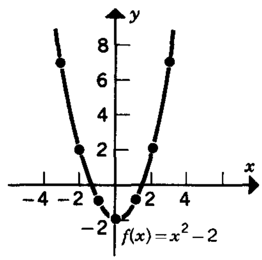

Figure 2 ■

When we graph a function, it is helpful to indicate the values of x for which the function is not defined. This procedure highlights regions of the xy-plane through which the graph does not pass. In addition, we can usually improve our sketch if we select values of x close to those that do not belong to the domain.

Example 7 Sketch the graph of the function $f(x) = \dfrac{3}{2 - x}$.

Solution First, note that the function is not defined when $x = 2$; this means that the graph will not contain a point whose x coordinate equals 2. The graph can be sketched by selecting values of x, taking care to include *nonintegral* values close to 2, and calculating the corresponding values of y. The resulting graph is shown in Figure 3.

x	$y = \dfrac{3}{2 - x}$
−4.00	0.50
−3.00	0.60
−2.00	0.75
−1.00	1.00
0.00	1.50
1.00	3.00
1.50	6.00
2.00	undefined
2.50	−6.00
3.00	−3.00
4.00	−1.50
5.00	−1.00

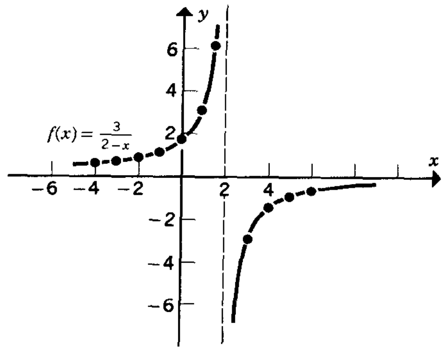

Figure 3

■

Not all equations in two variables represent functions. If the equation has the property that, for some x, there are two or more corresponding values of y, the equation does not represent a function. A simple illustration of this situation is the equation

$$y^2 = x - 1$$

Note: Except for $x = 1$, two values of y are paired with each value of x, for which a solution can be found. For example, when $x = 5$, two solutions for y are generated from the resulting equation $y^2 = 4$,

$$y_1 = +2 \qquad y_2 = -2$$

We can sketch the graph of the equation by first rewriting the equation as $y = \pm\sqrt{x - 1}$; because $x - 1 \geq 0$, or $x \geq 1$, the graph will not pass through points for which $x < 1$, as Figure 4 illustrates.

x	$y = \pm\sqrt{x - 1}$
1	0
2	±1
5	±2
10	±3

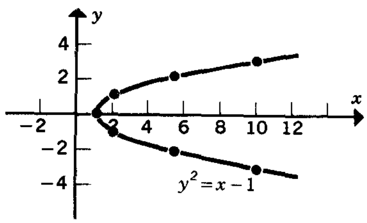

Figure 4

■

Vertical-Line Test

The graph of an equation in two variables can often indicate whether or not the equation represents a function. If a vertical line drawn on the coordinate system intersects the curve at two or more points, the curve cannot represent a function; the intersection of a vertical line with two or more points on the curve shows